

Homework 2 - Solutions

① IMPORTANT: $\frac{d}{d\theta} \hat{e}_r = \hat{e}_\theta$ AND $\frac{d}{d\theta} \hat{e}_\theta = -\hat{e}_r$

$$\frac{d\hat{e}_r}{dr} = \frac{d\hat{e}_\theta}{d\theta} = 0$$

If Φ is a SCALAR FIELD

$$\underline{\nabla} \Phi = \frac{\partial \Phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{e}_\theta$$

If \underline{u} is a VECTOR FIELD

$$\begin{aligned} \underline{\nabla} \underline{u} &= \frac{du_r}{dr} \hat{e}_r \otimes \hat{e}_r + \frac{du_\theta}{dr} \hat{e}_\theta \otimes \hat{e}_r + \frac{1}{r} \left(\frac{du_r}{d\theta} - u_\theta \right) \hat{e}_r \otimes \hat{e}_\theta \\ &\quad + \frac{1}{r} \left(\frac{du_\theta}{d\theta} - u_r \right) \hat{e}_\theta \otimes \hat{e}_\theta \end{aligned}$$

2) IT FOLLOWS FROM THE PRODUCT RULE INTRODUCED IN

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$$\begin{aligned} (b) \nabla(\phi \underline{u}) &= \pi(\underline{u}, \underline{D}\phi) + \pi(\underline{D}\underline{u}, \phi) = \\ &= \underline{u} \otimes \underline{D}\phi + \phi \underline{D}\underline{u} \end{aligned}$$

$$(a) \text{ as } \operatorname{div}(\cdot) = \operatorname{tr}(\nabla \cdot)$$

$$\begin{aligned} \text{From (b)} \quad \operatorname{tr}(\nabla(\phi \underline{u})) &= \operatorname{tr}(\underline{u} \otimes \underline{D}\phi) + \operatorname{tr}(\phi \underline{D}\underline{u}) = \\ &= \underline{u} \cdot \underline{D}\phi + \phi \operatorname{tr}(\underline{D}\underline{u}) = \underline{u} \cdot \underline{D}\phi + \phi \operatorname{div} \underline{u} \end{aligned}$$

(3) FROM THE DIVERGENCE THEOREM

FOR ALL CONSTANT VECTORS \underline{a} WE HAVE

$$\left[\int_{\partial R} (\underline{u} \otimes \underline{v}) dA \right] \underline{a} = \int_{\partial R} (\underline{a} \cdot \underline{v}) \underline{u} dA = \int_R \nabla(\underline{a} \cdot \underline{v}) dV =$$

$$\int_R \left\{ \underline{D}_i^T \underline{a} + \underline{D}_i^T \underline{v} \right\} dV = \left(\int_R \underline{D}_i^T dV \right) \underline{a} \quad \cancel{+ \underline{v}}$$

Then

$$\int_{\partial R} (\underline{m} \otimes \underline{v}) dA = \int_R \underline{\nabla} \underline{v}^T dv$$

$$\rightarrow \int_{\partial R} (\underline{v} \otimes \underline{m}) dA = \int_R \underline{\nabla} \underline{v} dv$$

(4) By applying the above result

$$\int_{\partial R} \underline{r} \otimes \underline{m} dA = \int_R \underline{\nabla} \underline{r} dv = \int_R \underline{I} dv = \underline{I} \int_R dv$$