CHAPTER 5 -

THE LINEAR ELASTICITY PROBLEM

AFTER EXPLORING, SOMEHOW INDEPENDENTLY, THE THREE MAIN ASPECTS OF SOLID MECHANICS (KINEMATICS, STRESS-EQUILIBRIUM AND MATERIAL BEHAVIOR) WE FINALLY WILL "GLUE THE PARTS" AND PUT ALTOGETHER AND END UP WITH A MATHEMATICAL MODEL DESCRIBING THE MECHANICAL RESPONSE OF A SOLID (DEFORMABLE) BODY WHEN SUBMITTED TO EXTERNAL LOADS.

PLEASE REMEMBER THAT SOME IMPORTANT HYPHOTESIS HAS BEEN ASSUMED (AND THEY WILL BE ALWAYS ADMITTED ALONG THIS COURSE), NAMELY: SMALL DEFORMATIONS; EQUILIBRIUM AND LINEAR ELASTIC RESPONSE.

THE MAIN RELATIONS ARE SUMMARIZED BELLOW:

(KINEMATICS)

$$\varepsilon = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$$

(EQUILIBRIUM)

$$\mathrm{div}\, T + \underline{b} = \underline{0}$$

$$T = T^T$$

(ELASTIC CONSTITUTIVE EQUATION - HOOKE'S LAW)

$$T = \lambda \, tr(\varepsilon) \, I + 2\mu \, \varepsilon$$

OR

$$\varepsilon = \frac{1+\nu}{E} T - \frac{\nu}{E} tr(T) \, I$$

REMARKS: (i) EACH ONE OF THE RELATIONS ABOVE IS OFTEN WRITTEN IN COMPONENTS. THAT WOULD BE A GOOD EXERCISE

(ii) THE LAMÉ PARAMETERS λ AND μ ARE RELATED TO THE YOUNG MODULUS E AND TO POISSON RATIO ν THROUGH

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{AND} \quad \mu = \frac{E}{2(1+\nu)}$$

(iii) THE BASIC RELATIONS CAN BE COMBINED IN DIFFERENT WAYS, GIVING RISE TO DIFFERENT (BUT EQUIVALENT) MATHEMATICAL PROBLEMS. THE MAIN DIFFERENCE RELIES ON WHAT IS THE SET OF PRIMARY VARIABLES. THE CHOICE AMONG THE POSSIBILITIES IS OFTEN MADE TARGETING A SOLUTION TECHNIQUE.

5.1) DISPLACEMENT FORMULATION

FIRST NOTE (NOW IN COMPONENTS)

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

REMEMBER THE CONVENTION

AND INSERTING THE ABOVE FORMULATION IN THE EQUILIBRIUM EQUATION WE OBTAIN:

$$\mu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) + \lambda \frac{\partial^2 u_k}{\partial x_k \partial x_j} \delta_{ij} + b_i = 0$$

$$\Longrightarrow \mu \frac{\partial^2 u_i}{\partial x_j^2} + (\mu + \lambda) \frac{\partial^2 u_k}{\partial x_k \partial x_i} + b_i = 0$$

OR

$$\mu \operatorname{div} (D \underline{u}) + (\lambda + \mu) \nabla (\operatorname{div} \underline{u}) + \underline{b} = \underline{0}$$

OR STILL

$$\boxed{\mu \, \Delta \underline{u} + (\lambda + \mu) \, \nabla (\mathrm{div}\, \underline{u}) + \underline{b} = 0}$$

WHERE Δ IS OFTEN REFFERED TO AS THE LAPLACIAN.

THE PARTIAL DIFFERENTIAL EQUATION, OFTEN CALLED NAVIER EQUATION(S), DOES NOT CONSTITUTE A WELL-POSED MATHEMATICAL PROBLEM, AS NO BOUNDARY CONDITIONS WERE SET. REMEMBER, AS SECOND DERIVATIVES ARE INVOLVED, BOUNDARY CONDITIONS CAN CONTAIN ZERO OR FIRST ORDER DERIVATIVES. FROM A PHYSICAL STANDPOINT, WE MISS ACTIONS GOING ON ALONG THE BOUNDARIES ... TYPICALLY WE CAN THINK EITHER ON INTERACTIONS BEING MODELED AS FORCES OR AS KINEMATIC RESTRICTIONS. IN THE FORMER CASE WE ARE SUPPOSED TO DO SOMETHING BEFORE INTRODUCING FORCES ACTING IN THE EXTERNAL SURFACE AS BOUNDARY CONDITIONS FOR THE NAVIER EQUATION (NOTE THAT THE EQUATION IS WRITTEN (e.g. HAS AS UNKNOWNS) ONLY DISPLACEMENTS).

SO, IF WE HAVE A PRESCRIBED FORCE DISTRIBUTION $\underline{s}^P$ ACTING ON A PORTION $\partial \mathbb{B}_\sigma$ OF THE BOUNDARY

$$T \underline{n} = \underline{s}^P \qquad x \in \partial \mathbb{B}_\sigma$$

AND THUS

$$\lambda \, tr(\varepsilon) \underline{n} + 2\mu \, \varepsilon \underline{n} = s^P \qquad x \in \partial \mathbb{B}_\sigma$$

WHICH INVOLVES FIRST-ORDER DERIVATIVES OF THE DISPLACEMENT FIELD. THIS IS ALWAYS REFFERED TO AS TRACTION BOUNDARY CONDITIONS.

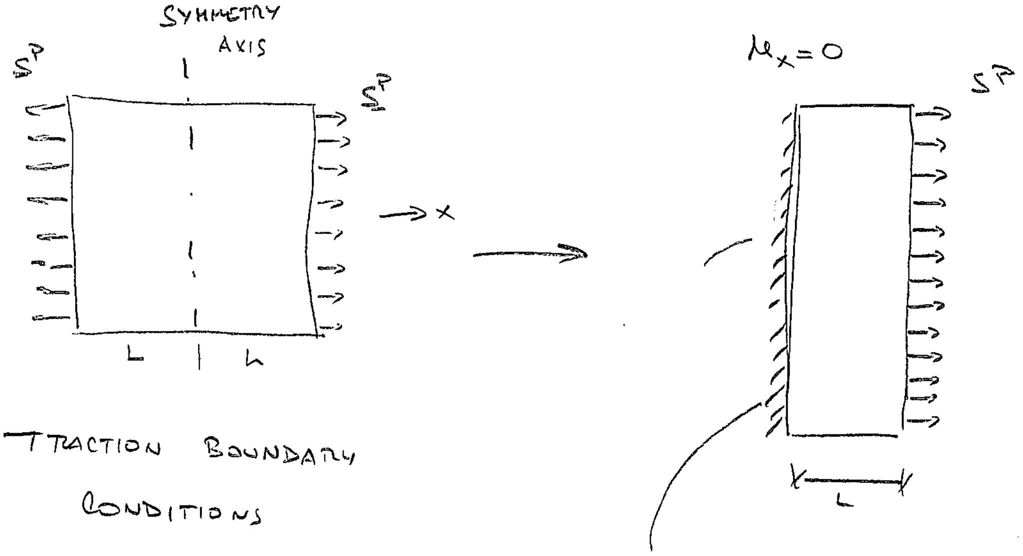
THE SECOND TYPE OF INTERACTION (EXPRESSED AS A BOUNDARY CONDITION TO THE NAVIER EQUATION) IS MODELED BY A KINEMATIC RESTRICTION LIKE

$$\underline{u} = \underbrace{\underline{u}^P}_{\text{KNOWN}} \qquad x \in \partial \mathbb{B}_u$$

WHICH INVOLVES A ZEROTH-ORDER DERIVATIVE OF THE DISPLACEMENT FIELD.

<u>QUESTION</u>: WHAT HAPPENS WITH FREE REGIONS OF THE BOUNDARY?

REMARK: OFTEN, BOUNDARY CONDITIONS ARE PRODUCED BY EXPLORING THE SYMMETRY OF THE PROBLEM, FOR INSTANCE



$\underbrace{T_{ex} \cdot e_y}_{\text{TANGENT COMPONENT}} = 0$ (WHY ?)

NOTE THAT IN THE REDUCED PROBLEM (AFTER APPLYING THE SYMMETRY CONSIDERATIONS) WE HAVE, AT THE SAME BOUNDARY A COMBINATION OF TRACTION AND DISPLACEMENT BOUNDARY CONDITIONS.

THE DISPLACEMENT FORMULATION LEADS TO A MATHEMATICAL PROBLEM WHICH HAS AS SOLUTION THE DISPLACEMENT FIELD. THE STRAIN FIELD CAN BE OBTAINED BY DIFFERENTIATION AND AFTER THE STRESS FIELD IS COMPUTED THROUGH THE CONSTITUTIVE EQUATIONS.

5.2) Stress Formulation

The idea is to eliminate $\underline{u}$ and ε and obtain a math. problem having T as the primal unknown. Departing from Hooke's law and using the compatibility equations

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial \varepsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial \varepsilon_{jl}}{\partial x_k \partial x_i} = 0$$

we obtain

$$\frac{\partial^2 \sigma_{ij}}{\partial x_k^2} + \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} - \frac{\partial^2 \sigma_{ik}}{\partial x_j \partial x_k} - \frac{\partial^2 \sigma_{jk}}{\partial x_i \partial x_k} =$$

$$\frac{\nu}{1+\nu} \left(\frac{\partial^2 \sigma_{mm}}{\partial^2 x_{kk}} \delta_{ij} + \frac{\partial^2 \sigma_{mm}}{\partial x_i \partial x_j} \delta_{kk} - \frac{\partial^2 \sigma_{mm}}{\partial x_j \partial x_k} \delta_{ik} \right.$$

$$\left. - \frac{\partial^2 \sigma_{mm}}{\partial x_i \partial x_k} \delta_{jk} \right)$$

AND NOW, APPLYING THE EQUILIBRIUM EQUATION

$$\left(\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \right)$$

$$\frac{\partial^2 \sigma_{ij}}{\partial x_k^2} + \frac{1}{1+\nu} \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = \frac{\nu}{1+\nu} \frac{\partial^2 \sigma_{mm}}{\partial x_k^2} \delta_{ij} - \frac{\partial b_i}{\partial x_j} - \frac{\partial b_i}{\partial x_i}$$

FOR THE CASE $i=j$

$$\frac{\partial^2 \sigma_{ii}}{\partial x_k^2} = - \frac{1+\nu}{1-\nu} \frac{\partial b_i}{\partial x_i}$$

AND SUBSTITUTING IT BACK IN THE PREVIOUS EQUATION

$$\frac{\partial^2 \sigma_{ij}}{\partial x_k^2} + \frac{1}{1+\nu} \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = - \frac{\nu}{1-\nu} \delta_{ij} \frac{\partial b_k}{\partial x_k} - \frac{\partial b_i}{\partial x_j} - \frac{\partial b_j}{\partial x_j}$$

THESE ARE THE BELTRAMI-MICHELL COMPATIBILITY EQUATIONS.

FOR THE CASE WITH NO BODY FORCES:

$$(1+\nu)\left[\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_x}{\partial z^2}\right]+\frac{\partial^2}{\partial x^2}\left[\sigma_x+\sigma_y+\sigma_z\right]=0$$

$$(1+\nu)\left[\frac{\partial^2\sigma_y}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial z^2}\right]+\frac{\partial^2}{\partial y^2}\left[\sigma_x+\sigma_y+\sigma_z\right]=0$$

$$(1+\nu)\left[\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right]+\frac{\partial^2}{\partial z^2}\left[\sigma_x+\sigma_y+\sigma_z\right]=0$$

$$(1+\nu)\left[\frac{\partial^2\sigma_{xy}}{\partial x^2}+\frac{\partial^2\sigma_{xy}}{\partial y^2}+\frac{\partial^2\sigma_{xy}}{\partial z^2}\right]+\frac{\partial^2}{\partial x\partial y}\left[\sigma_x+\sigma_y+\sigma_z\right]=0$$

$$(1+\nu)\left[\frac{\partial^2\sigma_{xz}}{\partial x^2}+\frac{\partial^2\sigma_{xz}}{\partial y^2}+\frac{\partial^2\sigma_{xz}}{\partial z^2}\right]+\frac{\partial^2}{\partial x\partial z}\left[\sigma_x+\sigma_y+\sigma_z\right]=0$$

$$(1+\nu)\left[\frac{\partial^2\sigma_{yz}}{\partial x^2}+\frac{\partial^2\sigma_{yz}}{\partial y^2}+\frac{\partial^2\sigma_{yz}}{\partial z^2}\right]+\frac{\partial^2}{\partial y\partial z}\left[\sigma_x+\sigma_y+\sigma_z\right]=0$$

IN THE ABOVE SET OF EQUATIONS ONLY THREE ARE INDEPENDENTS. THEREFORE, IN ORDER TO OBTAIN A MEANINGFUL MATHEMATICAL PROBLEM, THEY MUST BE COMBINED WITH THE EQUILIBRIUM EQUATION YIELDING A PROBLEM IN WHICH THE SIX-COMPONENTS OF T ARE THE UNKNOWNS AND THERE ARE ONLY TRACTION BOUNDARY CONDITIONS. ONCE THE STRESSES ARE OBTAINED, THE STRAIN FIELD CAN BE COMPUTED BY USING THE CONSTITUTIVE EQUATION. MOREOVER, THE DISPLACEMENT $\underline{u}$ (AND THEREFORE THE DEFORMED CONFIGURATION) IS CALCULATED BY INTEGRATING, WHICH DETERMINES IT UP TO AN ARBITRARY RIGID-BODY MOTION (MANY TIMES THIS CAN BE ELIMINATED BY OTHER CONSIDERATIONS ABOUT THE PROBLEM).

USUALLY, THE RESULTING PROBLEM FORMULATED IN TERMS OF STRESS COMPONENTS IS EXPLORED THROUGH THE USE OF "STRESS FUNCTIONS". THIS FORMALISM INTRODUCES FUNCTIONS THAT AUTOMATICALLY SATISFY THE EQUILIBRIUM

EXAMPLE.

A SIMPLE EXAMPLE

ASSUME THAT WE ARE DEALING WITH A PLANE STRESS PROBLEM ($\sigma_z = \sigma_{zy} = \sigma_{zx} = 0$;). THEN THE STRESS FORMULATION REDUCES TO

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = 0$$

!!

(HOW?)

IT IS ALSO TO BE NOTICED THAT THE PLANE STRESS HYPHOTHESIS INCLUDES $\sigma_x(x,y)$ AND $\sigma_y(x,y)$.

NOW THE STRATEGY RELIES ON ADMITING THAT EXISTS

A SCALAR FUNCTION ϕ (NAMED AIRY STRESS FUNCTION)

THAT SATISFIES $\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$; $\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$ AND $\sigma_{xy} = -\frac{\partial \phi}{\partial x \partial y}$.

IMMEDIATELY, WE RECOGNIZE THAT, FOR ANY ϕ, THE

EQUILIBRIUM EQUATIONS ARE SATISFIED. NOW, WE HAVE

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

WHICH IS THE BIHARMONIC EQUATION.

NOW CONSIDER THE FOLLOWING SPECIFIC EXAMPLES

(1) $\phi = a + b.x + cy$

$\longrightarrow$ IT SATISFIES THE BIHARMONIC EQUATION

AND

$\sigma_x = 0$; $\sigma_y = 0$ AND $\sigma_{xy} = 0$

RIGID BODY MOTION!

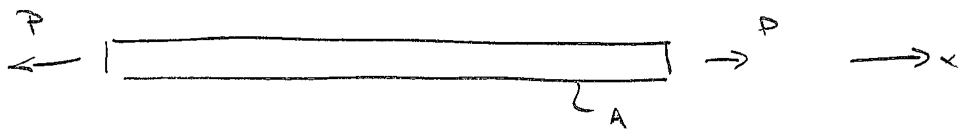
(ii) $\phi = a x^2 + bxy + cy^2$ (SATISFIES THE BIHARMONIC EQUATION)

$$\sigma_x = 2c \; ; \quad \sigma_y = 2a \; ; \quad \tau_{xy} = -b$$

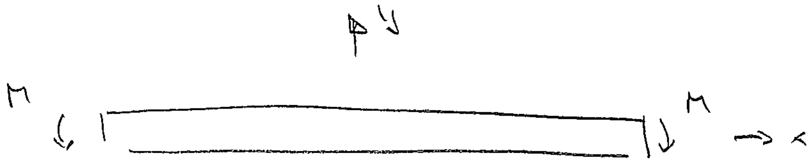
CONSTANT STRESS

IN THE CASE OF UNIAXIAL TENSION IN x DIRECTION

$$\phi = \frac{P}{2A} y^2$$



(iii) PURE BENDING



$$\sigma_x = \frac{My}{I} \, , \quad \tau_{xy} = 0 \quad \text{AND} \quad \sigma_{yy} = 0$$

$$\hookrightarrow \phi = \frac{M}{6I} y^3$$

<u>REMARK</u> : THE GOAL OF THE ABOVE EXAMPLES IS NOT SOLVING PROBLEMS BUT OFFERING THE FORM OF ϕ IN WELL KNOWN SITUATIONS.

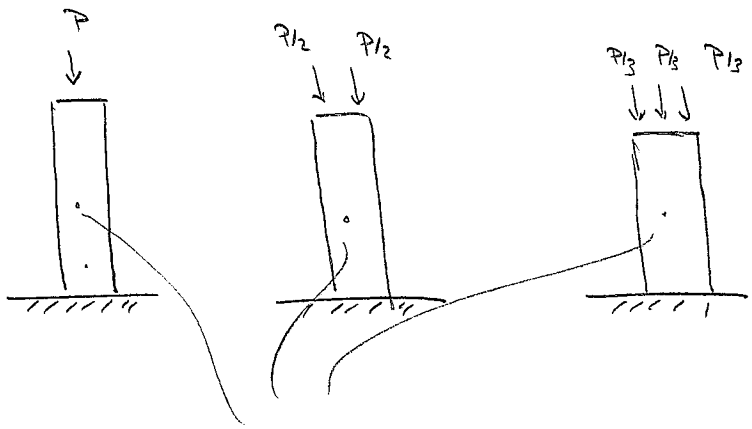
5.3- Examples and Solution Strategies

There is no generic way for solving an elasticity problem, although there are some very usefull and common tools and strategies. Those go from powerfull computational techniques (e.g.: finite element method, finite differences, boundary elements) to analytical approaches, like the stress functions we have just been introduced to. Computational methods are beyond the scope of the present course, so we will see a number of examples in which we don't have to resort on them.

It is worth to mention that, due to the linearity of the elasticity problem (here understood as obtaining the strain and stress fields across the body, with known initial geometry and material properties, when it is loaded by known external forces), the superposition principle plays a crucial role on solving actual problems.

Moreover, it is also important highlight that the analytical solutions, that will be soon presented, aditional "simplifying hypothesis are assumed. Among them, often uniform distributions are admited. This is often

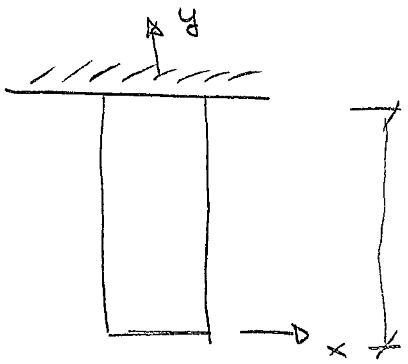
REASONABLE ONE WE ARE ANALYSING REGIONS OF THE BODY AWAY FROM LOADING OR SUPPORTIN AREA. THIS IS RELATED TO THE SO CALLED SAINT-VENAUT'S PRINCIPLE, WHICH IS ILLUSTRATED IN THE SKETCH BELLOW



WE EXPECT THAT IN THIS PART STRESS AND STRAINS FOR THE THREE DIFFERENT SITUATIONS ARE APPROXIMATELY THE SAME.

(WHAT ABOUT THE REGION WHERE LOADS ARE BEING APPLIED ?)

EXAMPLE 5.1: STRETCHING OF PRISMATIC BAR UNDER ITS OWN WEIGHT



$\underline{b} = (0, -\rho g, 0)$ (ρ BEING THE MATERIAL MASS DENSITY AND g THE ACCELERATION OF GRAVITY)

→ ADMITING THAT ON EACH SECTION OF THE BAR THE STRESS DISTRIBUTION IS UNIFORM, THEN (WHY?)

$$\sigma_z = \sigma_x = \sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0$$

THUS THE EQUILIBRIUM EQUATION REDUCES TO

$$\frac{\partial \sigma_y}{\partial y} = \rho g$$

AND

$$\sigma_y = \rho g y + \cancel{c_0} \quad (\sigma_y(0) = 0)$$

THEREFORE,

$$\varepsilon_y = \frac{\rho g}{E} Y$$

AND

$$\varepsilon_x = \varepsilon_z = -\frac{\nu \rho g}{E} Y$$

$$\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{xz} = 0$$

INTEGRATING THE ABOVE RELATIONS WE OBTAIN THE DISPLACEMENT FIELD AS

$$u_x = -\frac{\nu \rho g x Y}{E}$$

$$u_z = -\frac{\nu \rho g z Y}{E}$$

$$u_y = \frac{\rho g}{2E}\left[Y^2 + \nu(x^2+z^2) - L^2\right]$$

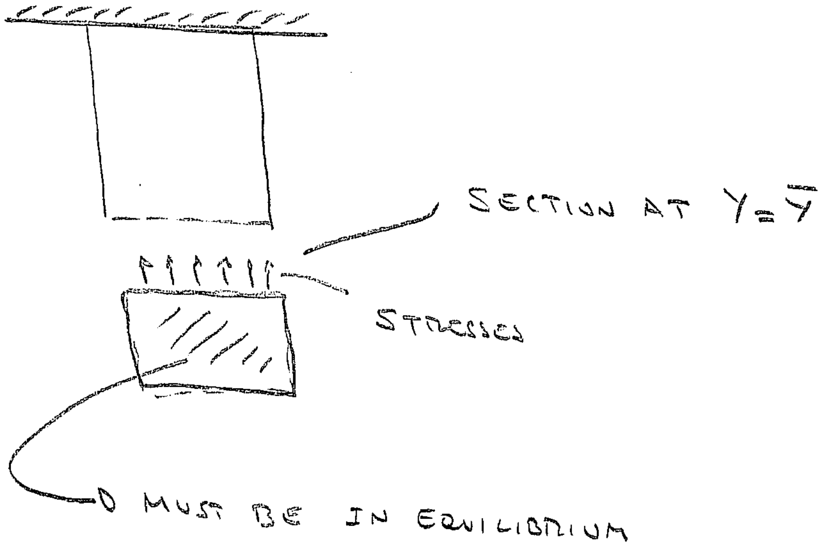
NOTE THAT WE HAVE USED THE BOUNDARY CONDITION

$$u_y(0, L, 0) = 0$$

AND

$$\frac{du_y}{dx}(0, L, 0) = \frac{du_y}{dz}(0, L, 0) = 0$$

LET US NOW LOOK AT EQUILIBRIUM AND STRESSES



SO: $\sigma_y = \rho g Y$; AND ALL OTHER COMPONENTS OF THE STRESS TENSOR ARE ZERO. PLEASE NOTE THAT AUTOMATICALLY SATISFIES THE EQUILIBRIUM EQUATION AND TRACTION BOUNDARY CONDITIONS! (DO YOU REALLY UNDERSTAND THIS POINT?)

FROM THE CONSTITUTIVE EQUATIONS (HOOKE'S LAW):

$$\varepsilon_y = \frac{\rho g Y}{E} \; ; \quad \varepsilon_x = -\nu \frac{\rho g Y}{E} \; ; \quad \varepsilon_z = -\nu \frac{\rho g Y}{E}$$

NOW ALL IS A MATTER OF INTEGRATING..., BUT REMEMBER YOU HAVE TO DEAL WITH THE CONSTANTS RELATED TO POSSIBLE RIGID BODY MOTIONS.

WE HAVE PASSED VERY FAST THROUGH THE EXERCISE. THUS, LET ANALYSE WITH MORE CARE THE DETAILS. FIRST NOTE THAT, AS MENTIONED BEFORE, WE HAVE NOT FOLLOWED ANY STANDARD RECIPE (LIKE WE OFTEN DO WHEN USING A FINITE ELEMENT CODE, FOR INSTANCE). THE BIG PICTURE IS: "OFFERING" A SOLUTION THAT SATISFIES THE EQUATIONS (ELASTICITY PROBLEM) BY EXPLORING REASONABLE MEANS. THIS SCENARIO LEADS US TO DIFFERENT PATHWAYS TO SOLVE THE PROBLEM.

A FIRST POINT TO BE NOTICED COMES TO SYMMETRY WITH RESPECT TO Y AXIS IN BOTH PLANES XY AND ZY (REMEMBER THIS IS A 3-D EXAMPLE). THIS SYMMETRY IMPLIES THAT:

$$u_x(x,y,z) = -u_x(-x,y,z)$$

$$u_z(x,y,z) = -u_z(x,y,-z)$$

$$u_y(x,y,z) = u_y(-x,y,z)$$

$$u_y(x,y,z) = u_y(x,y,-z)$$

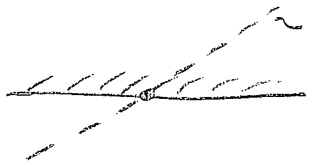
$$\vdots \qquad \vdots$$

SO, TO HANDLE THIS ISSUE WE HAVE TO UNDERSTAND BETTER THE BOUNDARY CONDITIONS. NOTE THAT THE ONLY KINEMATICAL RESTRICTIONS TO BE APPLIED ARE RESTRICTED TO THE $Y=L$ SECTION. THIS IS THE MORE DELICATE POINT OF THE PROBLEM, AS YOU WOULD PROBABLY THINK OF $\underline{u}(x,y,z)=\underline{0}$. NOTE THAT ADOPTING THIS B.C THE DEFORMATION FIELD AT $Y=L$ WOULD BE INCOMPATIBLE WITH THE PROPOSED STRESS FIELD (WHY?). THIS LEADS (IF YOU WILL TO MAINTAIN A CERTAIN DEGREE OF SIMPLICITY IN THE SOLUTION...) TO "MODELING": $u_y(0,L,0)=0$

AND NO ROTATIONS ARE ALLOWED AT THAT POINT WHICH IMPLIES $\dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} = 0$ AND $\dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} = 0$

(WE WILL COME TO THESE EXPRESSIONS LATER)

GEOMETRICALLY



SECTION ROTATION IS NOT ALLOWED!

THEN:

$$u_x = -\frac{\nu \rho g}{E} YX + C(Y,Z)$$

$\hookrightarrow$ CONSTANT ON X

FROM SYMMETRY $C(Y,Z) = 0$ (WHY?)

ANALOGOUS $u_z = -\frac{\nu \rho g}{E} YZ$

FINALY:

$$u_y = \frac{\rho g Y^2}{2E} + D(X,Z)$$

BUT $u_y(0,0,0) = 0 \longrightarrow D(0,0) \neq 0$

AND

$$\frac{\partial D}{\partial X}(0,0) = 0$$

$$\frac{\partial D}{\partial Z}(0,0) = 0$$

SO A GOOD GUESS FOR D WOULD BE

$$D = k(X^M + Z^M)$$ (WITH M EVEN DUE TO THE SYMMETRY). BUT NOTE THAT FROM THE DEFORMATION

$$\frac{\partial D}{\partial x} = -\frac{\nu \rho g X}{E} \longrightarrow \boxed{M = 2} \text{ AND } \boxed{k = \nu}$$

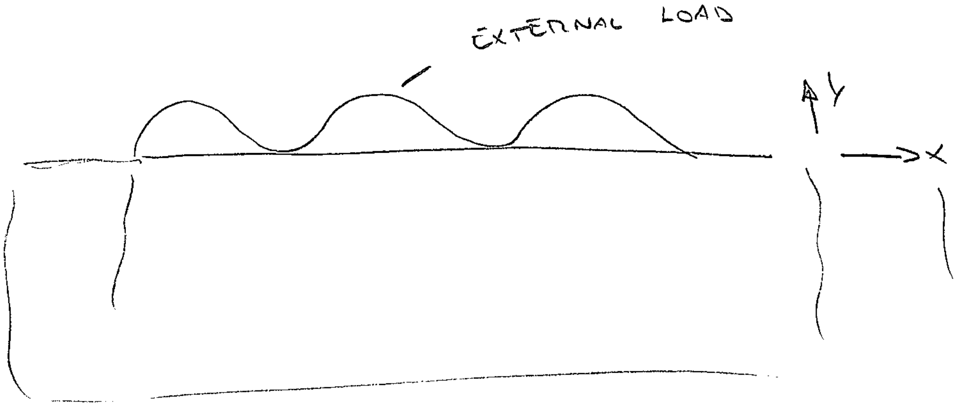
REMARK: ROTATIONS IN THE CONTEXT OF SMALL DEFORMATIONS. AS ANY OTHER TENSOR, THE GRADIENT OF DEFORMATION CAN BE DECOMPOSED INTO A SYMMETRICAL AND AN ANTI-SYMMETRICAL PARTS, NAMELY:

$$\nabla \underline{u} = \underline{\varepsilon} + \underline{W}$$

WHERE: $\underline{W} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$ (COMPONENTS: $W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$)

IS THE ROTATION TENSOR AND IT IS EASY FIGURE OUT (FROM GEOMETRICAL CONSIDERATIONS) THAT, FOR INSTANCE, W_{13} MAY BE INTERPRETED AS THE AVERAGE ROTATION AROUND X_2.

EXAMPLE 5.2: CONSIDER THE SEMI-INFINITE BODY DESCRIBED BY THE HALF-PLANE $Y<0$ LOADED BY A SURFACE DISTRIBUTION PERIODIC FORCE $\underline{S} = \tau_0 \cos wx \, \underline{e_x} + \sigma_0 \sin wx$



THEREFORE, ON THE SURFACE $Y=0$

$$T_{\underline{M}} = \underline{S} \rightarrow \sigma_{yy} = \sigma_0 \sin wx \quad \text{AND} \quad \tau_{xy} = \sigma_0 \cos wx$$

WHICH SUGEST THE FOLLOWING FORM TO THE AIRY STRESS FUNCTION

$$\phi = f(y) \sin wx \quad \text{(REMEMBER THE DEFINITION OF } \phi\text{!)}$$

THEN, PLUGING THE ABOVE FUNCTION INTO THE BI-HARMONIC EQUATION

$$\frac{\partial^4}{\partial y^4} f \sin wx + 2 \frac{\partial^2 f}{\partial y^2} \left(-w^2 \sin wx\right) + f \left(w^4 \sin wx\right) = 0$$

$$\rightarrow \frac{\partial^4 f}{\partial y^4} - 2w^2 \frac{\partial^2 f}{\partial y^2} + w^4 f = 0$$

TRYING $f = f_0 e^{\lambda y} \rightarrow \lambda^4 - 2\lambda^2 w^2 + w^4 = 0$

$$\lambda \begin{cases} w \\ -w \end{cases}$$

AS THE COMPONENT STRESSES ARE FINITE $\lambda > 0$

AND ϕ CAN ASSUME THE FORM

$$\phi = \left(C_1 e^{wy} + C_2 y e^{wy}\right) \sin wx$$

APPLYING THE BOUNDARY CONDITIONS:

$$y = 0 \rightarrow \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = -C_1 w^2 \sin wx = \sigma_0 \sin wx$$

$$\rightarrow \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \left(-C_1 w + C_2\right) w \cos wx = \tau_0 \cos wx$$

THUS

$$C_1 = -\frac{\sigma_0}{w^2}, \quad C_2 = \frac{-\tau_0}{w} - C_1 w = \frac{\sigma_0 - \tau_0}{w}$$