

# Métodos de Energia

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**Ref. e ilustrações : An Introduction to the Mechanics of Solids . Crandall et. All. – 2.6, 5.8, 6.8, 7.8**



## Introdução : considerações gerais

- Balanço de energia : uma visão alternativa
- Sólidos constituídos por materiais elásticos :  
conservação de energia (1ª lei da Termodinâmica)
- Princípio dos Trabalhos (Potências) Virtuais : fornece a base para desenvolvimento de métodos para simulação computacional (Elementos Finitos)

## Introdução : balanço de energia

Um exemplo : o sistema massa mola

Balanço de Momento Linea ("Equilíbrio")

$$F - Kx = m\ddot{x}$$

$$x(0) = x_0 \quad \dot{x}(0) = \dot{x}_0$$



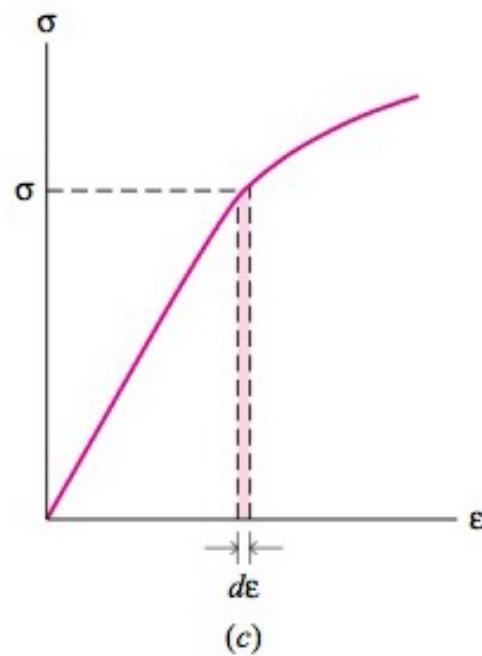
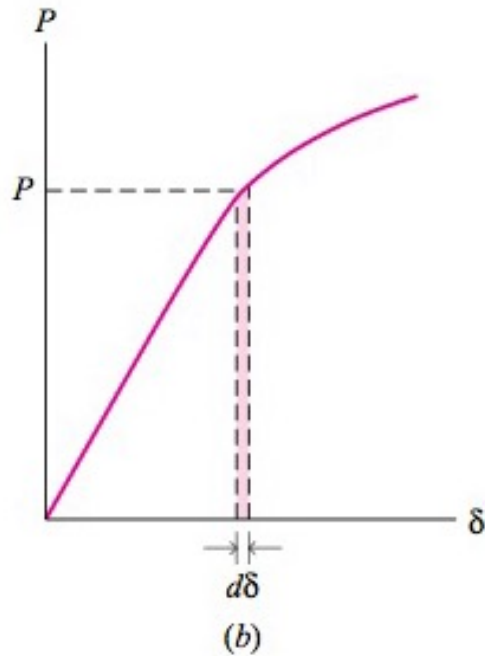
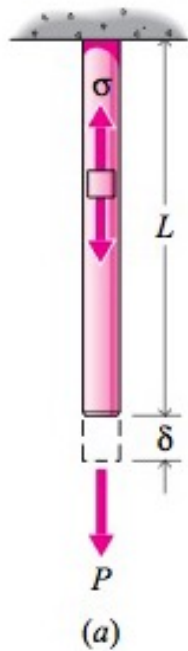
$$F \dot{x} - Kx \dot{x} = m\ddot{x} \dot{x}$$

Energia Cinética:  $E = \frac{1}{2}m\dot{x}^2$     Energia Potencial (elástica) :  $V = \frac{1}{2}kx^2$

$$\underbrace{F\dot{x}}_{\text{pot. externa}} = \frac{d}{dt} \overbrace{(E + V)}^{T \text{ energia do sistema}}$$

$$T(t) - T(0) = \int_0^t F \dot{x} dt = \underbrace{\int_{x_0}^x F dx}_{\text{trabalho da força externa}}$$

# Introdução : o cenário undimensional elástico linear



$$\sigma = \frac{P}{A} ; \epsilon = \frac{\sigma}{E} ; \epsilon = \frac{\delta}{L}$$

Trabalho da força externa

$$W = \int_0^{\delta_2} P d\delta = \int_0^{\delta_2} \sigma A d\delta = \int_0^{\epsilon_2} \sigma AL d\epsilon$$

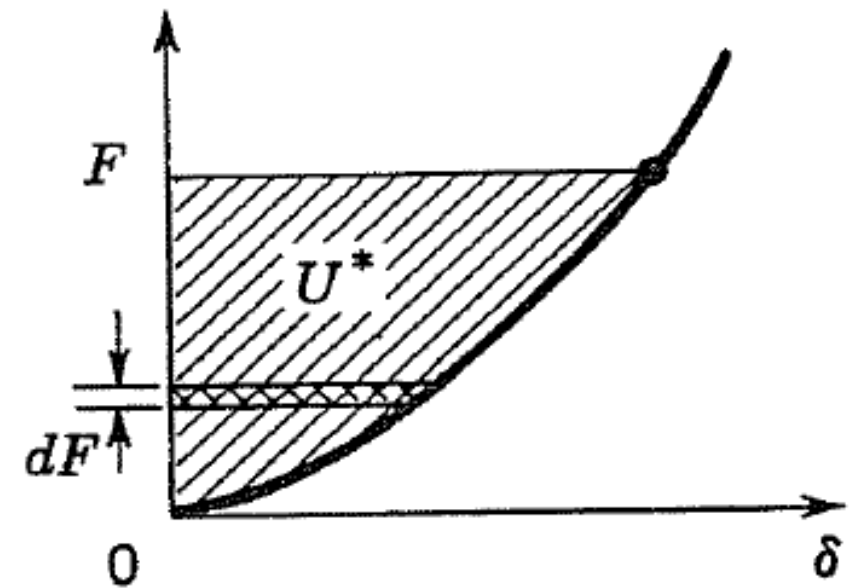
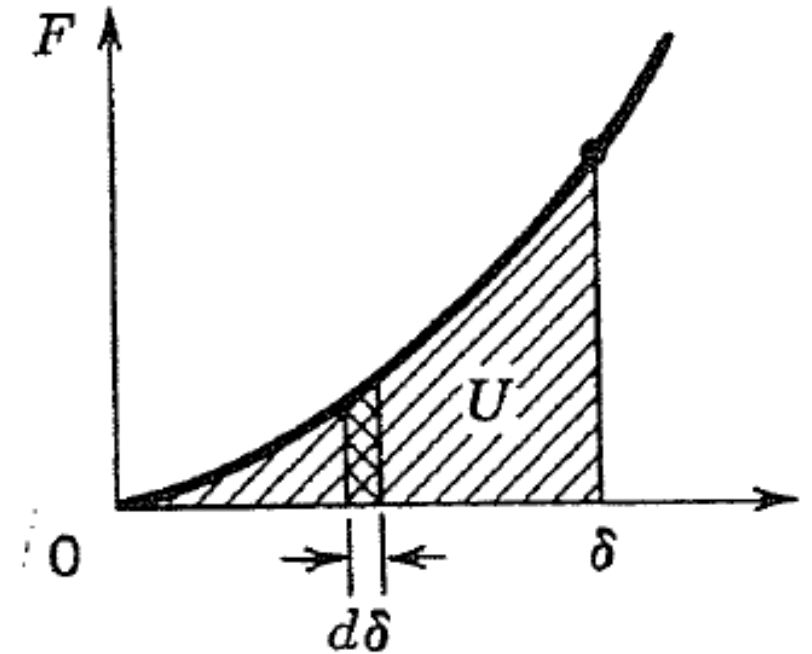
Balanço de energia

$$U = W = \frac{\sigma^2 AL}{2E} = \frac{P^2 L}{2EA}$$

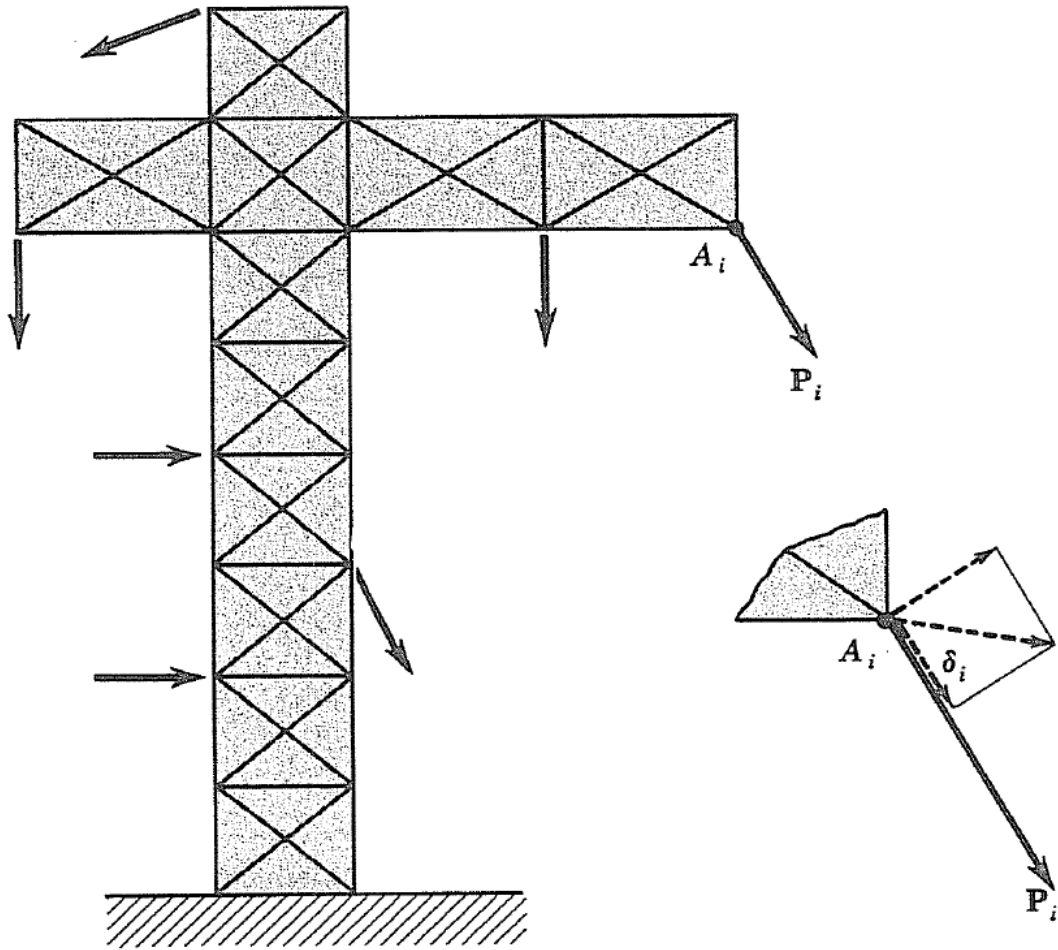
$$U^* = \frac{\epsilon^2 ALE}{2} = \frac{\delta^2 AE}{2L}$$

Generalizando ...

$$U = \int_0^{\delta} P. d\delta ; U^* = \int_0^P \delta. dP$$



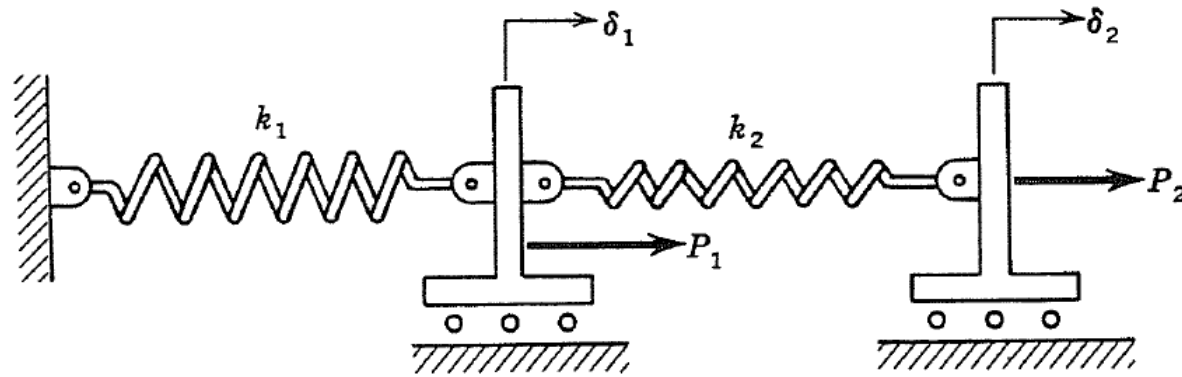
# O Teorema de Castigliano e suas aplicações



$$\sum_i \int_0^{P_i} \mathbf{s}_i \cdot d\mathbf{P}_i = \sum_i \int_0^{P_i} \delta_i \cdot dP_i = U^*$$

$$\frac{\partial U^*}{\partial P_i} = \delta_i$$

# Exemplos..



$$V = \frac{1}{2}kx^2 = \frac{1}{2}k_{xx} = \frac{1}{2}Fx = \frac{1}{2}F \frac{F}{k} = \frac{F^2}{2k}$$

$$V^* = V$$

equilíbrio

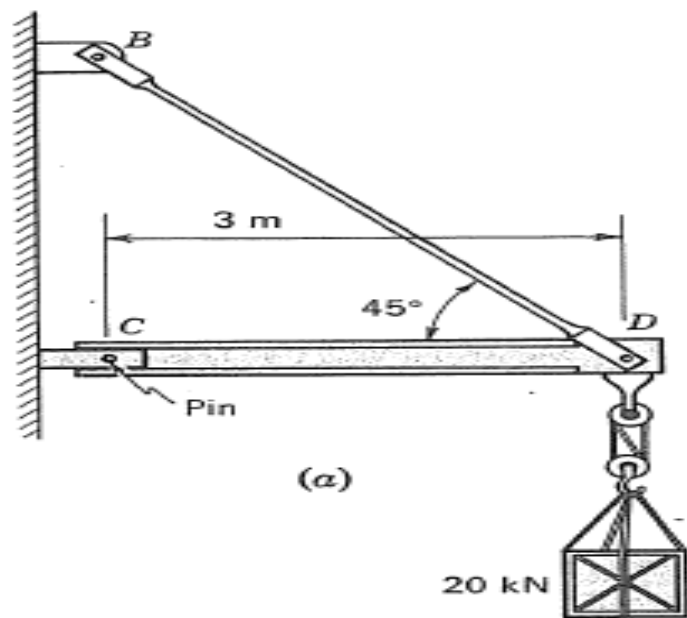
$$F_1 = P_1 + P_2$$

$$F_2 = P_2$$

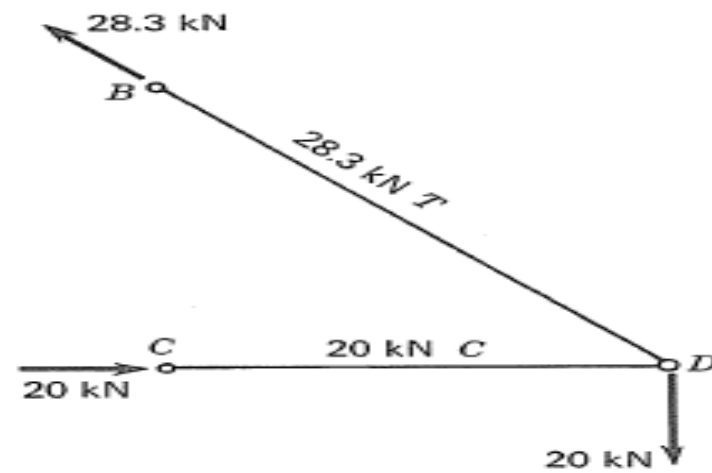
$$U = U_1 + U_2 = \frac{(P_1 + P_2)^2}{2k_1} + \frac{P_2^2}{2k_2}$$

$$\delta_1 = \frac{\partial U}{\partial P_1} = \frac{P_1 + P_2}{k_1}$$

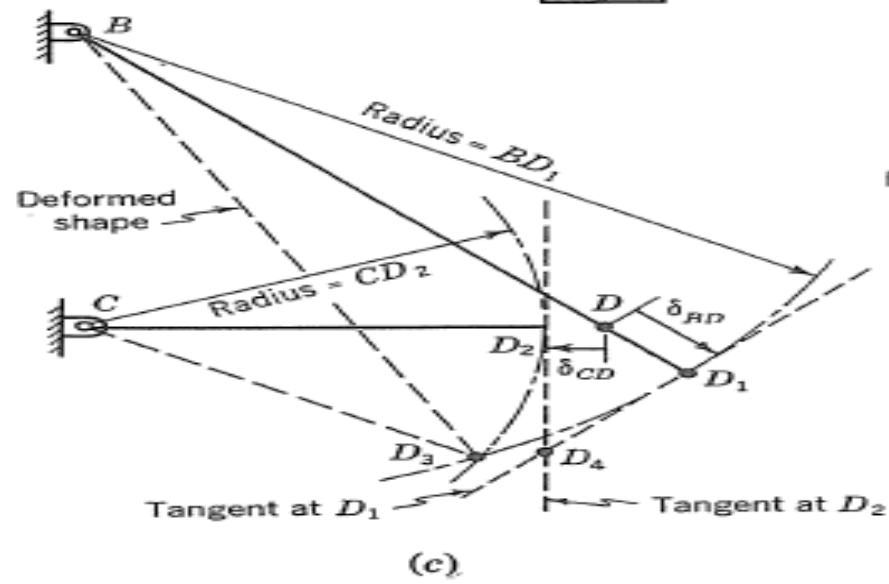
$$\delta_2 = \frac{\partial U}{\partial P_2} = \frac{P_1 + P_2}{k_1} + \frac{P_2}{k_2}$$



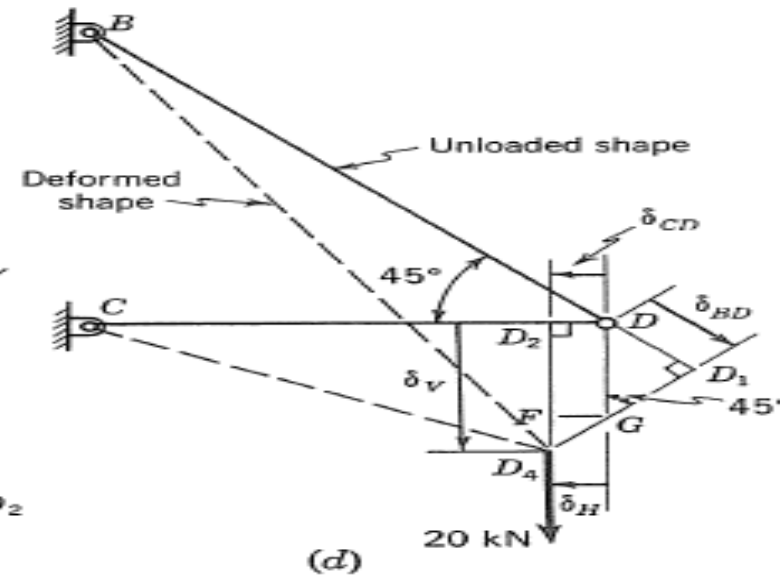
(a)



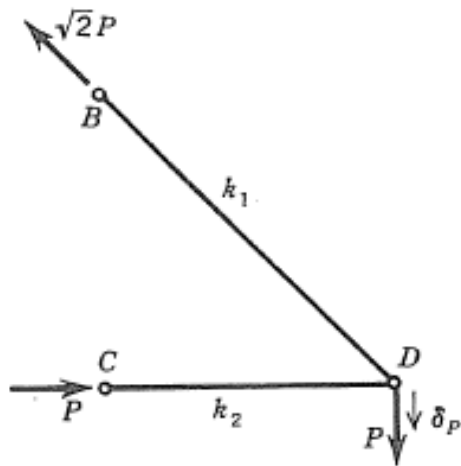
(b)



(c)



(d)



$$k_1 = \frac{491 \times 10^{-6} \times 205 \times 10^6}{3 \sqrt{2}}$$

$$= 23.73 \text{ MN/m}$$

$$k_2 = \frac{3.2 \times 10^{-3} \times 205 \times 10^6}{3}$$

$$= 218.67 \text{ MN/m}$$

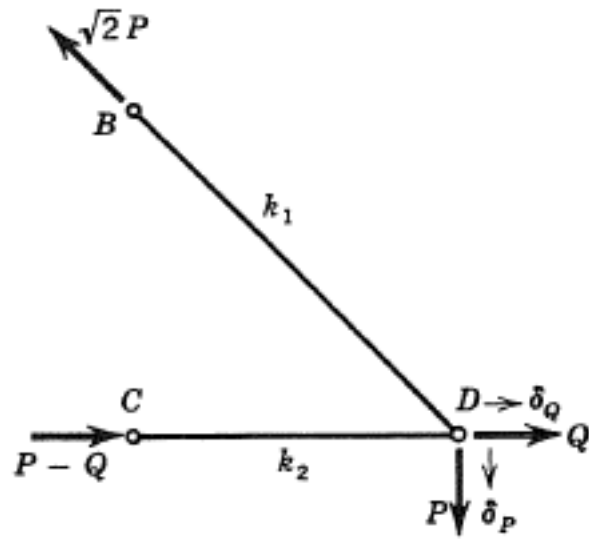
$$U = U_1 + U_2 = \frac{P_1^2}{2k_1} + \frac{P_2^2}{2k_2} = \frac{2P^2}{2k_1} + \frac{P^2}{2k_2}$$

$$k_i = \frac{A_i E_i}{L_i} \quad ???$$

$$\delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \left( \frac{P^2}{k_1} + \frac{P^2}{2k_2} \right) = 2P \left( \frac{1}{k_1} + \frac{1}{2k_2} \right)$$

$$\delta_P = 2 \times 20 [0.0421 + 0.0023] \times 10^{-6} = 1.77 \text{ mm}$$

Olhando na solução apresentada anteriormente, nota-se que o deslocamento do ponto D tem componentes na direção vertical (como já calculado) mas também na direção horizontal. No entanto, a força aplicada é na vertical.... O que fazer?

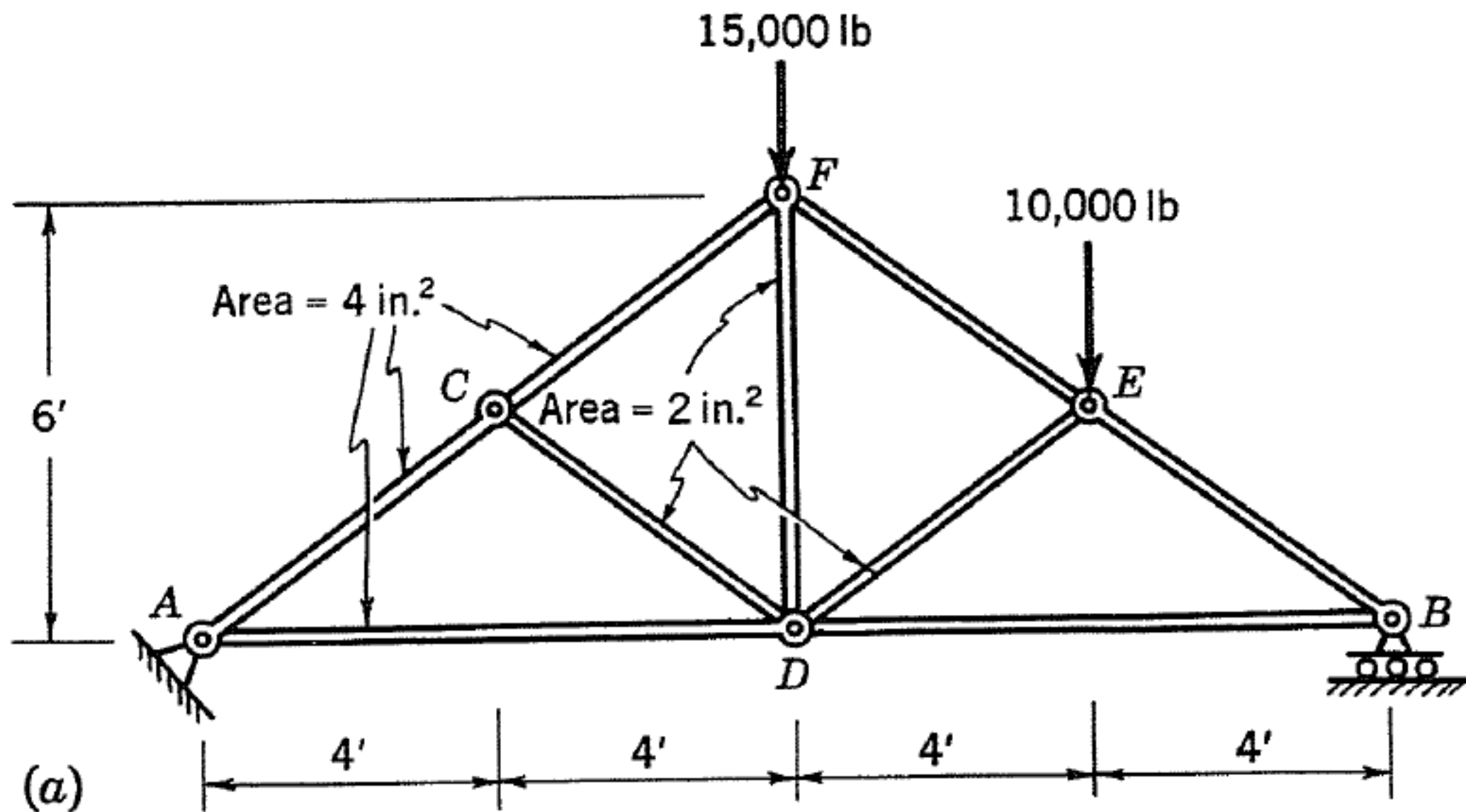


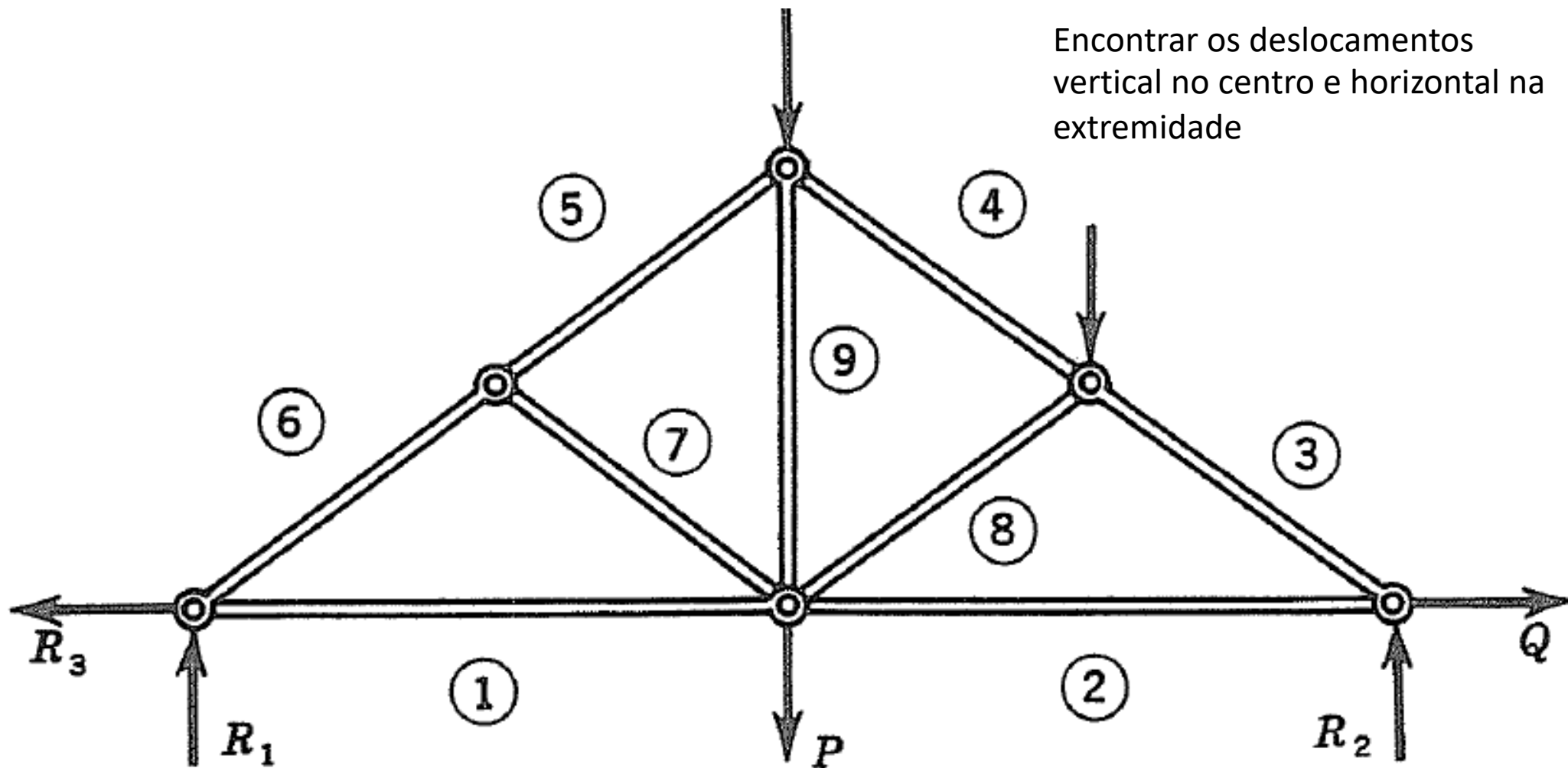
and

$$U = \frac{2P^2}{2k_1} + \frac{1}{2k_2} (P - Q)^2$$

$$\delta_Q = \frac{\partial U}{\partial Q} = 0 - \frac{P - Q}{k_2}$$

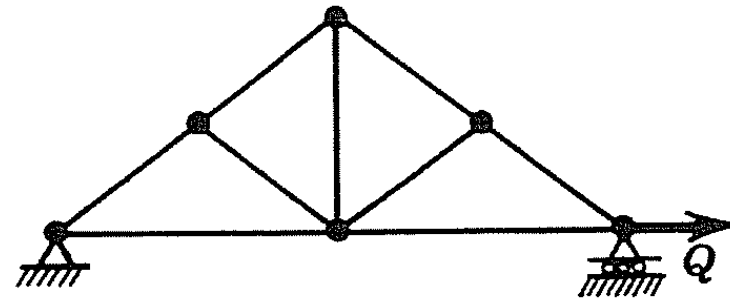
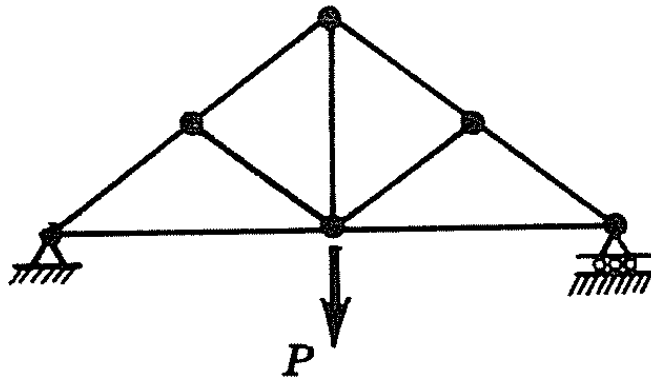
$$\delta_Q = \frac{-P}{k_2} = -0.0915 \text{ mm}$$





$$U_i = \frac{F_i^2 L_i}{2A_i E_i} \quad \delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \sum_{i=1}^n \frac{F_i^2 L_i}{2A_i E_i} = \sum_{i=1}^n \frac{F_i L_i}{A_i E_i} \frac{\partial F_i}{\partial P}$$

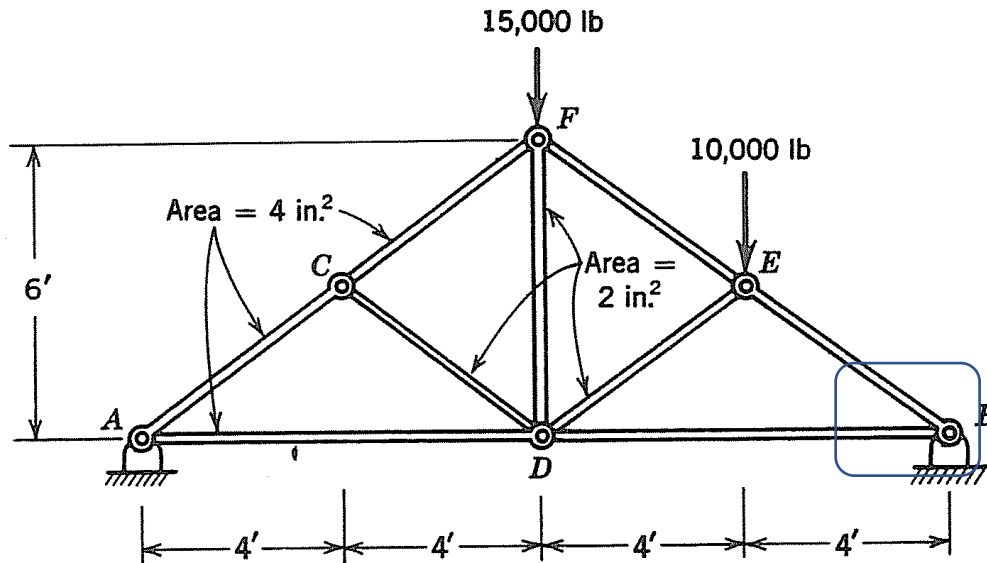
$$\delta_P = \sum_{i=1}^n F_i \frac{L_i}{A_i E_i} \frac{\partial F_i}{\partial P}$$



**Table 2.6 Truss solution by energy methods**

| $i$ | $F_i$<br>$10^3 \text{ lb}$ | $(L/AE)^*$<br>$\text{in./lb}$ | $\frac{\partial F_i}{\partial P}$ | $\frac{\partial F_i}{\partial Q}$ | $\left(\frac{FL}{AE} \frac{\partial F}{\partial P}\right)_i^\dagger$ | $\left(\frac{FL}{AE} \frac{\partial F}{\partial Q}\right)_i^\dagger$ |
|-----|----------------------------|-------------------------------|-----------------------------------|-----------------------------------|--|--|
| 1   | $+13.33 + Q$               | $2.4 \times 10^{-6}$          | $+\frac{2}{3}$                    | $+1$                              | $21.36 \times 10^{-3}$   | $32.0 \times 10^{-3}$  |
| 2   | $+20.0 + Q$                | $2.4 \times 10^{-6}$          | $+\frac{2}{3}$                    | $+1$                              | $31.95 \times 10^{-3}$   | $48.0 \times 10^{-3}$  |
| 3   | $-25.0$                    | $1.5 \times 10^{-6}$          | $-\frac{5}{6}$                    | 0                                 | $31.26 \times 10^{-3}$   |  |
| 4   | $-16.67$                   | $1.5 \times 10^{-6}$          | $-\frac{5}{6}$                    | 0                                 | $20.85 \times 10^{-3}$   |  |
| 5   | $-16.67$                   | $1.5 \times 10^{-6}$          | $-\frac{5}{6}$                    | 0                                 | $20.85 \times 10^{-3}$   |  |
| 6   | $-16.67$                   | $1.5 \times 10^{-6}$          | $-\frac{5}{6}$                    | 0                                 | $20.85 \times 10^{-3}$   |  |
| 7   | 0                          | $1.5 \times 10^{-6}$          | 0                                 | 0                                 | 0  |  |
| 8   | $-8.33$                    | $3.0 \times 10^{-6}$          | 0                                 | 0                                 | 0  |  |
| 9   | $+5.0$                     | $3.6 \times 10^{-6}$          | $+1$                              | 0                                 | $18.00 \times 10^{-3}$   |  |
|     |                            |                               |                                   |                                   | $\Sigma = 0.1651 \text{ in.}$<br>$= \delta_y$                        | $\Sigma = 0.080 \text{ in.}$<br>$= \delta_x$                         |

Um outro cenário (estaticamente não determinado)



Retomando o caso anterior e impondo

$$\frac{\partial U}{\partial Q} = 0$$

$$\sum F_i \frac{L_i}{A_i E_i} \frac{\partial F_i}{\partial Q} = 0 = [13.33 \times 10^3 + Q + 20 \times 10^3 + Q][2.4 \times 10^{-6}]$$

$$Q = -16.67 \times 10^3 \text{ lb}$$