Optimal Control of Dynamic Systems

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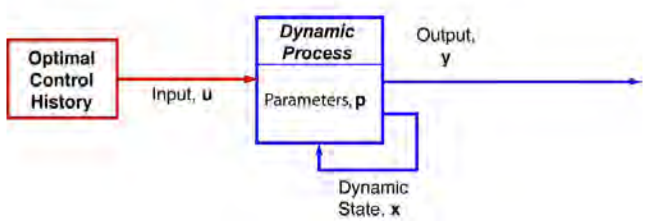
Optimal Control and Estimation, MAE 546,
Princeton University, 2018

- Dynamic systems
- Cost functions
- Problems of Lagrange, Mayer, and Bolza
- Necessary conditions for optimality
  - Euler-Lagrange equations
- Sufficient conditions for optimality
  - Convexity, normality, and uniqueness

The Dynamic Process



- *Dynamic Process*
  - *Neglect disturbance effects, w(t)*
  - *Subsume p(t) and explicit dependence on t in the definition of f[.]*

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)]$$

Trajectory of the System

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]$$

Integrate the dynamic equation to determine the trajectory from original time, t_o, to final time, t_f

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{f}[\mathbf{x}(\tau),\mathbf{u}(\tau)]\,d\tau$$

given $\mathbf{u}(t)$ for $t_0 \le t$

What Cost Function Might Be Minimized?

- **Minimize time required to go from *A* to *B***

$$J = \int_0^{\text{final time}} dt = \textbf{Final time}$$

- **Minimize fuel used to go from *A* to *B***

$$J = \int_0^{\text{final range}} (\text{Fuel-use Efficiency})\,dR = \textbf{Fuel Used}$$

- **Minimize financial cost of producing a product**

$$J = \int_0^{\text{final time}} (\text{Cost per hour})\,dt = \textbf{\$\$}$$

Optimal System Regulation

Minimize mean-square state deviations over a time interval

Scalar variation of a single component

$$J = \frac{1}{T}\int_0^T x^2(t)\, dt$$

dim(*x*) = 1 x 1

Sum of variation of all state elements

$$J = \frac{1}{T}\int_0^T \left[\mathbf{x}^T(t)\mathbf{x}(t)\right] dt = \frac{1}{T}\int_0^T \left[x_1^2 + x_1^2 + \cdots + x_n^2\right] dt$$

dim(**x**) = *n* x 1

Weighted sum of state element variations

$$J = \frac{1}{T}\int_0^T \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)\right] dt = \frac{1}{T}\int_0^T \left\{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right\} dt$$

n = 3
dim(**x**) = *n* x 1
dim(**Q**) = *n* x *n*

Why not use infinite control?

Tradeoffs Between State and Control Variations

Trade performance, x, against control usage, u

$$J = \int_0^T \left[x^2(t) + ru^2(t)\right] dt, \quad r > 0$$

dim(*u*) = 1 x 1

Minimize a cost function that contains state and control vectors

$$J = \int_0^T \left[\mathbf{x}^T(t)\mathbf{x}(t) + r\mathbf{u}^T(t)\mathbf{u}(t)\right] dt, \quad r > 0$$

dim(**u**) = *m* x 1

Weight the relative importance of state and control components

$$J = \int_0^T \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right] dt, \quad \mathbf{Q}, \mathbf{R} > 0$$

dim(**R**) = *m* x *m*

Classical Cost Functions for Optimizing Dynamic Systems

The Problem of Lagrange

(*c. 1780*)

$$\min_{\mathbf{u}(t)} J = \int_{t_0}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right]dt$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_o) \textit{ given}$$

$$\dim(\mathbf{x}) = n \times 1$$
$$\dim(\mathbf{f}) = n \times 1$$
$$\dim(\mathbf{u}) = m \times 1$$

The integrand, *L*[x(t), u(t)], is called the Lagrangian

$L\left[\mathbf{x}(t),\mathbf{u}(t)\right] = \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]$ **Quadratic trade between state and control**

$= 1$ **Minimum time problem**

$= \dot{m}(t) = fcn\left[\mathbf{x}(t),\mathbf{u}(t)\right]$ **Minimum fuel use problem**

$L\left[\mathbf{x}(s),\mathbf{u}(s)\right] =$ Change in area with respect to differential length, e.g., fencing, ds [Maximize]

The Problem of Mayer
(*c. 1890*)

$$\boxed{\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right]}$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_o)\ given$$

Examples of Terminal Cost

$$\phi\left[\mathbf{x}(t_f)\right] = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)\Big|_{t=t_f}$$ **Weighted square - error in final state**

$$= \left|\left(t_{final} - t_{initial}\right)\right|$$ **Minimum time problem**

$$= \left|\left(m_{initial} - m_{final}\right)\right|$$ **Minimum fuel problem**

The Problem of Bolza (*c. 1900*)
The Modern Optimal Control Problem*

Combine the Problems of Lagrange and Mayer

- **Minimize the sum of terminal and integral costs**
  - By choice of $\mathbf{u}(t)$
  - Subject to dynamic constraint

$$\boxed{\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_0}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right] dt}$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_0)\ given$$

and <u>with fixed end time,</u> t_f

Augmented Cost Function

Adjoin dynamic constraint to integrand using a Lagrange multiplier to form the Augmented Cost Function, J_A:

$$J_A = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_0}^{t_f} \left\{ L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \underline{\boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right]} \right\} dt$$

$$\dim\left[\boldsymbol{\lambda}(t)\right] = \dim\left\{\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),t\right]\right\} = n \times 1$$

The Dynamic Constraint

$$\dim\left\{\boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right]\right\} = (1 \times n)(n \times 1) = 1$$

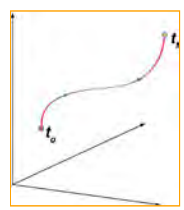
The constraint = 0 when the dynamic equation is satisfied

$$\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right] = 0 \text{ when } \dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] \text{ in } \left[t_0, t_f\right]$$

* **Lagrange multiplier is also called**
- **Adjoint vector**
- **Costate vector**

Necessary Conditions for a Minimum

Necessary Conditions for a Minimum



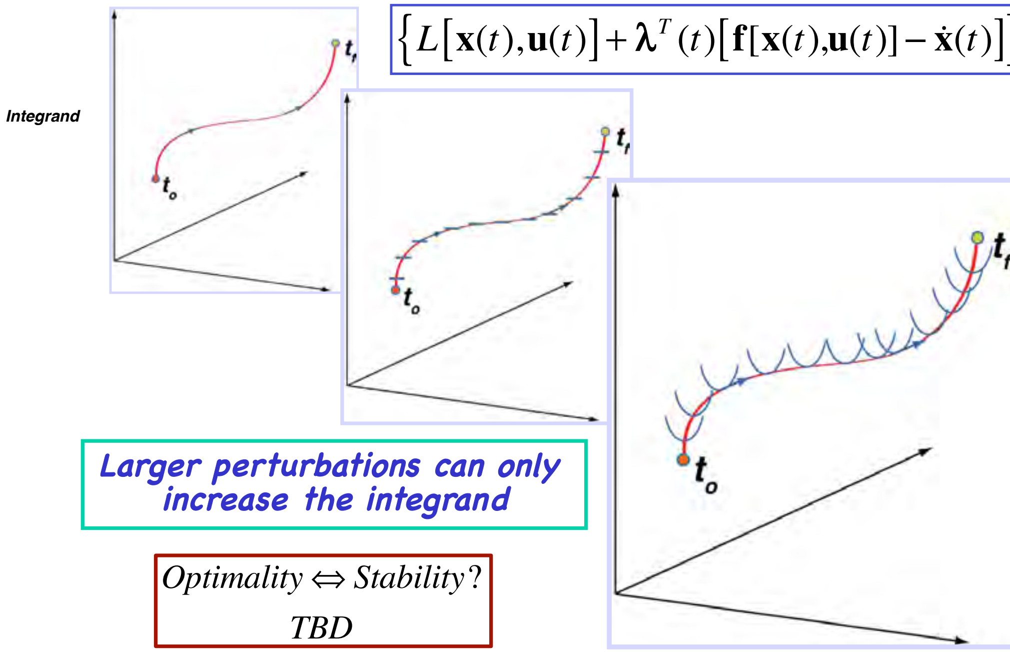
- Satisfy *necessary conditions for stationarity* along entire trajectory, from t_0 to t_f

- For integral to be minimized, integrand takes lowest possible value at every time
  - Linear insensitivity to small control-induced perturbations
  - Large perturbations can only increase the integral cost

- Cost is insensitive to control-induced perturbations occurring <u>at</u> the final time, t_f

Integrand of Cost Function must be linearly insensitive to control-induced perturbation

$$\left\{ L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right]\right\}$$



Larger perturbations can only increase the integrand

$$Optimality \Leftrightarrow Stability?$$

TBD

The Hamiltonian

Re-phrase the integrand by introducing the Hamiltonian

$$H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] = L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$

The Hamiltonian is a function of the Lagrangian, adjoint vector, and system dynamics

Integrand of the augmented cost function

$$\left\{ L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right]\right\} = \left\{ H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t)\right\}$$

Incorporate the Hamiltonian in the Cost Function

- **Variables in the Hamiltonian reflect**
  - integral cost
  - constraining effect of system dynamics
- **Substitute the Hamiltonian in the cost function**

$$\boxed{J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_0}^{t_f} \left\{ H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t) \right\} dt}$$

- **The optimal cost, J^*, is produced by the optimal histories of state, control, and Lagrange multiplier: $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$, and $\lambda^*(t)$**

$$\boxed{\min_{\mathbf{u}(t)} J = J^* = \phi\left[\mathbf{x}^*(t_f)\right] + \int_{t_0}^{t_f} \left\{ H\left[\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)\right] - \boldsymbol{\lambda}^{*T}(t)\dot{\mathbf{x}}^*(t) \right\} dt}$$

Integration by Parts

Scalar indefinite integral

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Vector definite integral

$$u = \boldsymbol{\lambda}^T(t)$$

$$dv = \dot{\mathbf{x}}(t)dt = d\mathbf{x}$$

Apply to second term in the integrand

$$\boxed{\int_{t_0}^{t_f} \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t)\,dt = \boldsymbol{\lambda}^T(t)\mathbf{x}(t)\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}^T(t)\mathbf{x}(t)\,dt}$$

Integrate the Cost Function By Parts

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_0}^{t_f} \left\{ H\left[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t) \right\} dt$$

$$u = \boldsymbol{\lambda}^T(t)$$
$$dv = \dot{\mathbf{x}}(t)dt = d\mathbf{x}$$

Cost function can be re-written as

$$J = \phi\left[\mathbf{x}(t_f)\right] + \left[\boldsymbol{\lambda}^T(t_0)\mathbf{x}(t_0) - \boldsymbol{\lambda}^T(t_f)\mathbf{x}_f(t)\right]$$
$$+ \int_{t_0}^{t_f} \left\{ H\left[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)\right] + \dot{\boldsymbol{\lambda}}^T(t)\mathbf{x}(t) \right\} dt$$

First-Order Variations

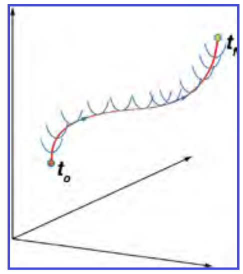
First variations in a quantity induced by control variations

$$\Delta(.) = \frac{\partial(.)}{\partial \mathbf{u}}\Delta\mathbf{u} + \frac{\partial(.)}{\partial \mathbf{x}}\Delta\mathbf{x}(\Delta\mathbf{u}) + \frac{\partial(.)}{\partial \boldsymbol{\lambda}}\Delta\boldsymbol{\lambda}(\Delta\mathbf{u})$$
$$= \frac{\partial(.)}{\partial \mathbf{u}}\Delta\mathbf{u} + \frac{\partial(.)}{\partial \mathbf{x}}\Delta\mathbf{x}(\Delta\mathbf{u}) + \frac{\partial(.)}{\partial \boldsymbol{\lambda}}(0)$$

$$\boxed{\Delta(.) = \frac{\partial(.)}{\partial \mathbf{u}}\Delta\mathbf{u} + \frac{\partial(.)}{\partial \mathbf{x}}\Delta\mathbf{x}(\Delta\mathbf{u})}$$

(The adjoint vector is a function of time alone)

Stationarity of the Cost Function



Cost must be insensitive to small variations in control policy along the optimal trajectory

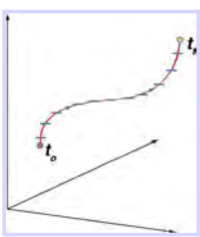
First variation of the cost function due to control

$$\Delta J^* = \left\{\left[\frac{\partial \phi}{\partial x} - \boldsymbol{\lambda}^T\right]\right\}\Delta \mathbf{x}(\Delta \mathbf{u})\bigg|_{t=t_f} + \left[\boldsymbol{\lambda}^T \Delta \mathbf{x}(\Delta \mathbf{u})\right]\Big|_{t=t_0} + \int_{t_0}^{t_f}\left\{\frac{\partial H}{\partial \mathbf{u}}\Delta \mathbf{u} + \left[\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right]\Delta \mathbf{x}(\Delta \mathbf{u})\right\}dt = 0$$
$$\equiv \Delta J(t_f) + \Delta J(t_0) + \Delta J(t_0 \rightarrow t_f)$$

Three, independent, necessary conditions for stationarity (Euler-Lagrange equations)

$$\boxed{\Delta J^* = 0}$$

First-Order Insensitivity to Control Perturbations



Individual terms of ΔJ^* must remain zero for arbitrary variations in $\Delta \mathbf{u}(t)$

$$1)\quad \left[\frac{\partial \phi}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T\right]\bigg|_{t=t_f} = \mathbf{0}$$

$\dot{\mathbf{x}}(0) = \mathbf{f}\left[\mathbf{x}(0), \mathbf{u}(0)\right]$ need not be zero, but

$\mathbf{x}(0)$ cannot change instantaneously unless control is infinite

$$\therefore \left[\Delta \mathbf{x}(\Delta \mathbf{u})\right]\Big|_{t=t_0} \equiv 0, \text{ so } \Delta J\big|_{t=0} = 0$$

$$2)\quad \left[\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right] = \mathbf{0} \quad in\left(t_0, t_f\right)$$

$$3)\quad \frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} \quad in\left(t_0, t_f\right)$$

Euler-Lagrange Equations

Euler-Lagrange Equations

Boundary condition for adjoint vector

$$1)\quad \boldsymbol{\lambda}(t_f) = \left\{ \frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

Ordinary differential equation for adjoint vector

$$2)\quad \dot{\boldsymbol{\lambda}}(t) = -\left\{ \frac{\partial H[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t]}{\partial \mathbf{x}} \right\}^T$$

$$= -\left[\frac{\partial L}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T \triangleq -\left[L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t) \right]^T$$

Jacobian matrices

$$\mathbf{F}(t) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t)$$

$$\mathbf{G}(t) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(t)$$

Optimality condition

$$3)\quad \frac{\partial H[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t]}{\partial \mathbf{u}} = \left[\frac{\partial L}{\partial \mathbf{u}} + \boldsymbol{\lambda}^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right] \triangleq \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t) \right] = \mathbf{0}$$

Jacobian Matrices

Jacobian Matrices Express Solution Sensitivity to Small Perturbations

Stability matrix

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{array}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$\dim(\mathbf{F}) = n \times n$

Control effect matrix

$$\mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{array}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$\dim(\mathbf{G}) = n \times m$

Jacobian Matrix Example

Original nonlinear equation describes nominal dynamics

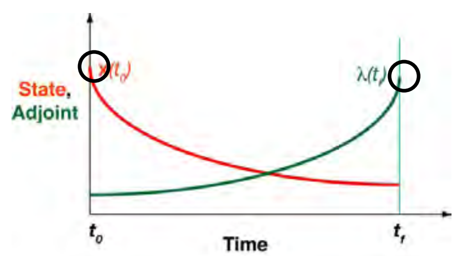
$$\dot{\mathbf{x}}_N(t) = \begin{bmatrix} \dot{x}_{1_N}(t) \\ \dot{x}_{2_N}(t) \\ \dot{x}_{3_N}(t) \end{bmatrix} = \begin{bmatrix} x_{2_N}(t) \\ a_2\left[x_{3_N}(t) - x_{2_N}(t)\right] + a_1\left[x_{3_N}(t) - x_{1_N}(t)\right]^2 + b_1 u_{1_N}(t) \ + b_2 u_{2_N}(t) \\ c_2 x_{3_N}(t)^3 + c_1\left[x_{1_N}(t) + x_{2_N}(t)\right] + b_3 x_{1_N}(t) u_{1_N}(t) \end{bmatrix}$$

Jacobian matrices are time-varying in the example

$$\mathbf{F}(t) = \left[\begin{array}{c:c:c} 0 & 1 & 0 \\ \hdashline -2a_1\left[x_{3_N}(t) - x_{1_N}(t)\right] & -a_2 & a_2 + 2a_1\left[x_{3_N}(t) - x_{1_N}(t)\right] \\ \hdashline \left[c_1 + b_3 u_{1_N}(t)\right] & c_1 & 3c_2 x_{3_N}^2(t) \end{array}\right]$$

$$\mathbf{G}(t) = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \\ b_3 x_{1_N}(t) & 0 \end{bmatrix}$$

Dynamic Optimization is a Two-Point Boundary Value Problem



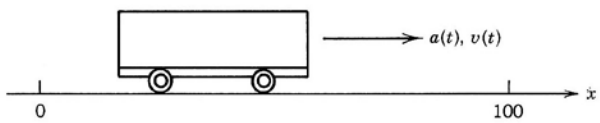
Boundary condition for the state equation is specified at t_0

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad \mathbf{x}(t_0) \ given$$

Boundary condition for the adjoint equation is specified at t_f

$$\dot{\boldsymbol{\lambda}}(t) = -\left[\frac{\partial L}{\partial \mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t)\right]^T, \quad \boldsymbol{\lambda}(t_f) = \left\{\frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}}\right\}^T$$

Sample Two-Point Boundary Value Problem
Move Cart 100 Meters in 10 Seconds



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \textit{Position} \\ \textit{Velocity} \end{bmatrix}$$

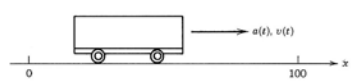
- **Cost function: tradeoff between**
  - **Terminal error squared**
  - **Integral cost of control squared**

$$J = q\left(x_{1_f} - 100\right)^2 + \int_{t_0}^{t_f} r u^2 \, dt$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}; \quad L = ru^2; \quad \phi = q\left(x_{1_f} - 100\right)^2$$

$$H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}] = L[\mathbf{x},\mathbf{u}] + \boldsymbol{\lambda}^T \mathbf{f}[\mathbf{x},\mathbf{u}]$$
$$= ru^2 + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

Closed-Form Solution for Adjoint Vector



$$\dot{\boldsymbol{\lambda}}(t) = -\left\{\frac{\partial H}{\partial \mathbf{x}}\right\}^T = -\left[\frac{\partial L}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]^T = -\left[0 + \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]^T$$

$$\boldsymbol{\lambda}(t_f) = \left\{\frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}}\right\}^T = \begin{bmatrix} 2q\left(x_{1_f} - 100\right) & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ \lambda_1 \end{bmatrix}; \quad \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_{t=t_f} = \begin{bmatrix} 2q\left(x_{1_f} - 100\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1(t_f) \\ \lambda_1(t_f)\left(t_f - t\right) \end{bmatrix} = \begin{bmatrix} 2q\left(x_{1_f} - 100\right) \\ 2q\left(x_{1_f} - 100\right)\left(t_f - t\right) \end{bmatrix}$$

Closed-Form Solution for Control History

Optimality condition

$$\left(\frac{\partial H}{\partial \mathbf{u}}\right)^T = \left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)^T \boldsymbol{\lambda}(t)\right] = \mathbf{0}$$

Optimal control strategy

$$2ru(t) + \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{bmatrix} 2q\left(x_{1_f} - 100\right) \\ 2q\left(x_{1_f} - 100\right)\left(t_f - t\right) \end{bmatrix} = 0$$

$$u(t) = -\left(\frac{q}{r}\right)\left(x_{1_f} - 100\right)\left(t_f - t\right) \triangleq k_1 + k_2 t$$

Cost Weighting Effects on Optimal Solution

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,dt, \quad t_0 \to t_f$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} k_1 t^2/2 + k_2 t^3/6 \\ k_1 t + k_2 t^2/2 \end{bmatrix}$$

$$u(t) = -\left(\frac{q}{r}\right)\left(x_{1_f} - 100\right)\left(t_f - t\right) \triangleq k_1 + k_2 t$$

For $t = 10s,\ x_{1_f} = \dfrac{100}{1 + 0.003\left(\dfrac{r}{q}\right)}$

q	100	1	1
r	1	1	100
k_1	3.000	2.991	2.308
k_2	−0.300	−0.299	−0.231
x_{1_f}	99.997	99.701	76.923
x_{2_f}	15.000	14.955	11.538
$\int u^2\,dt$	29.998	29.821	17.751
J	32.794	29.923	2307.7

Iterate to Find Optimal Trajectory for More Complex Problems

Calculate x(t) using prior estimate of u(t) , i.e., starting guess

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,dt, \quad t_0 \to t_f$$

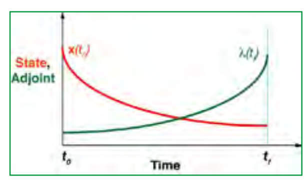
Evaluate Lagrangian in [t_0, t_f]

$$L\left[\mathbf{x}(t),\mathbf{u}(t)\right], \quad \text{in } \left(t_0, t_f\right)$$

Calculate adjoint vector using prior estimate of x(t) and u(t)

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(t_f) - \int_{t_f}^{t} \left[\frac{\partial L\left[\mathbf{x}(t),\mathbf{u}(t)\right]}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T(t) \frac{\partial \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]}{\partial \mathbf{x}} \right]^T dt, \quad t_f \to t_0$$

Typical Iteration to Find Optimal Trajectory



Calculate $H(t)$ and $\partial H/\partial$u using prior estimates of state, control, and adjoint vector

$$H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] = L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$

$$\frac{\partial H}{\partial \mathbf{u}} = \left[\frac{\partial L}{\partial \mathbf{u}} + \boldsymbol{\lambda}^T(t) \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right], \quad \text{in } \left(t_0, t_f\right)$$

Estimate new u(t)

$$\mathbf{u}_{new}(t) = \mathbf{u}_{old}(t) + \Delta\mathbf{u}\left\{ \frac{\partial H(t)}{\partial \mathbf{u}} \right\}, \quad \text{in } \left(t_0, t_f\right)$$

Alternative Necessary Condition for Time-Invariant Problem

Time-Invariant Optimization Problem

Time-invariant problem: Neither *L* nor *f* is explicitly dependent on time

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t] = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}]}$$

$$\boxed{L\left[\mathbf{x}(t),\mathbf{u}(t),t\right] = L\left[\mathbf{x}(t),\mathbf{u}(t)\right]}$$

Then, the Hamiltonian is

$$\boxed{\begin{aligned} H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\right] &= L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \\ &= H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] \end{aligned}}$$

Time-Rate-of-Change of the Hamiltonian for Time-Invariant System

$$\frac{dH[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)]}{dt} = \cancel{\frac{\partial H}{\partial t}} + \frac{\partial H}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial H}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial H}{\partial \boldsymbol{\lambda}}\frac{\partial \boldsymbol{\lambda}}{\partial t}$$

H **is independent of time**

$$\frac{dH}{dt} = \left[L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t)\right]\dot{\mathbf{x}} + \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right]\dot{\mathbf{u}} + \left[\mathbf{f}^T\right]\dot{\boldsymbol{\lambda}}$$

from Euler-Lagrange Equations #2 and #3

$$\frac{dH}{dt} = \left[\left(L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t)\right) + \dot{\boldsymbol{\lambda}}^{\mathbf{T}}\right]\dot{\mathbf{x}} + \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right]\dot{\mathbf{u}}$$
$$= [\mathbf{0}]\dot{\mathbf{x}} + [\mathbf{0}]\dot{\mathbf{u}} = \mathbf{0}\ \textbf{on optimal trajectory}$$

Hamiltonian is Constant on the Optimal Trajectory

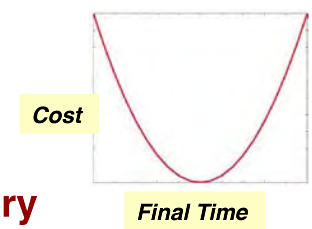
For time-invariant system dynamics and Lagrangian

$$\frac{dH}{dt} = 0 \Rightarrow H^* = \text{ constant on optimal trajectory}$$

H* **= constant is an alternative scalar necessary condition for optimality**

Open-End-Time Optimization Problem

Open End-Time Problem



Final time, t_f, is free to vary

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_0}^{\boxed{t_f}} \left\{ H\left[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t) \right\} dt$$

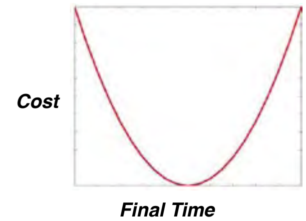
t_f is an <u>additional control variable</u> for minimizing J

$$\Delta J = \Delta J(t_f) + \Delta J(t_0) + \Delta J(t_0 \to t_f)$$

$$\Delta J(t_f) = \Delta J(t_f)\Big|_{fixed\ t_f} + \frac{dJ}{dt}\Big|_{t=t_f} \Delta t_f$$

Goal: t_f for which sensitivity to perturbation in final time is zero

Additional Necessary Condition for Open End-Time Problem



Cost sensitivity to final time should be zero

$$\left.\frac{dJ}{dt}\right|_{t=t_f} = \left.\left\{\left[\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial \mathbf{x}}\dot{\mathbf{x}}\right] + \left[H - \boldsymbol{\lambda}^T\dot{\mathbf{x}}\right]\right\}\right|_{t=t_f}$$

$$= \left.\left\{\left[\frac{\partial\phi}{\partial t} + H\right] + \left[\boldsymbol{\lambda}^T\dot{\mathbf{x}} - \boldsymbol{\lambda}^T\dot{\mathbf{x}}\right]\right\}\right|_{t=t_f}$$

$$= \left.\left\{\frac{\partial\phi}{\partial t} + H\right\}\right|_{t=t_f} = 0$$

Additional necessary condition

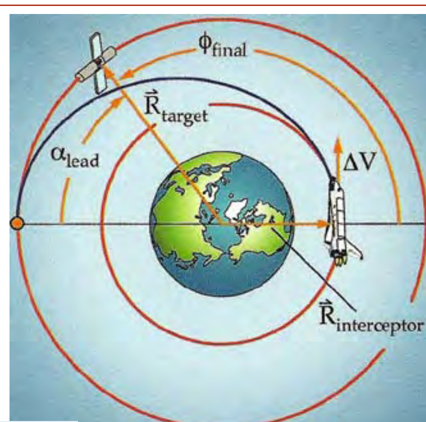
$$\boxed{\frac{\partial\phi^*}{\partial t} = -H^* \text{ at } t = t_f \quad \text{for open end time}}$$

Optimal Rendezvous Requires Phasing of the Maneuver

International Space Station is a moving target

Transfer orbit time equals target's time to reach rendezvous point

$$\phi\left[\mathbf{x}_{shuttle}\left(t_f\right)\right] = \frac{1}{2}\left[\mathbf{x}_{shuttle}\left(t_f\right) - \mathbf{x}_{ISS}\left(t_f\right)\right]^T \mathbf{P}\left[\mathbf{x}_{shuttle}\left(t_f\right) - \mathbf{x}_{ISS}\left(t_f\right)\right]$$



Sellers, 2005

H^* = 0 with Open End-Time and Time-Independent Terminal Cost

If terminal cost is independent of time, and final time is open

$$\left.\frac{dJ}{dt}\right|_{t=t_f} = \left.\left\{\frac{\partial\phi}{\partial t} + H\right\}\right|_{t=t_f} = \left.\left\{(0) + H\right\}\right|_{t=t_f} = 0$$

Hamiltonian at final time:

$$\therefore H^*\left(t_f\right) = 0$$

Hamiltonian for Time-Invariant System, Terminal Cost, and Open End Time

Time-invariant system

$$\frac{dH}{dt} = 0 \Rightarrow$$
$$H^* = \textbf{ constant } \text{in}\left[t_0, t_f\right]$$

Open end time

$$\frac{\partial\phi^*}{\partial t}\left(t_f\right) = -H^*\left(t_f\right)$$

Time-independent terminal cost

$$H^*\left(t_f\right) = 0$$

Therefore

$$H^* = 0 \text{ in}\left[t_0, t_f\right]$$

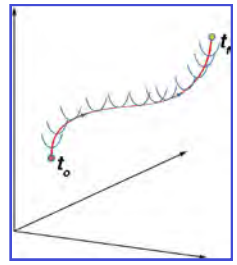
Sufficient Conditions for a Minimum

Sufficient Conditions for a Local Minimum

- **Euler-Lagrange equations are satisfied (necessary conditions for stationarity), plus proof of**
  - **Convexity**
  - **Controllability <--> Normality**
  - **Uniqueness**
- Singular optimal control
  - Higher-order conditions

Convexity

Legendre-Clebsch Condition



"Strengthened" condition

$$\frac{\partial^2 H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)}{\partial \mathbf{u}^2} > 0 \text{ in } (t_0, t_f)$$

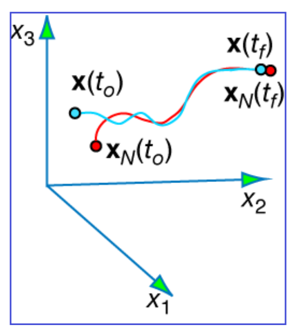
Positive definite (m x m) Hessian matrix throughout trajectory

"Weakened" condition

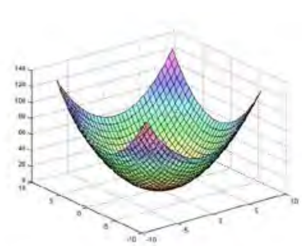
$$\frac{\partial^2 H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)}{\partial \mathbf{u}^2} \geq 0 \text{ in } (t_0, t_f)$$

Hessian may equal zero at isolated points

Normality and Controllability



- **Normality: Existence of <u>neighboring-optimal solutions</u>**
  - **Neighboring vs. neighboring-optimal trajectories**
- **Controllability: Ability to satisfy a <u>terminal equality constraint</u>**
- **Legendre-Clebsch condition satisfied**



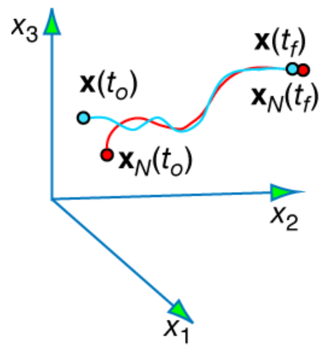
Neighboring vs. Neighboring-Optimal Trajectories

- **Nominal (or reference) trajectory and control history**

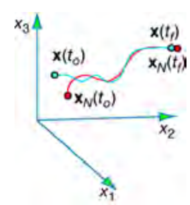
$$\left\{\mathbf{x}_N(t), \mathbf{u}_N(t)\right\} \quad \text{for } t \text{ in } \left(t_0, t_f\right)$$

- **Neighboring trajectory**
  - Small initial condition variation
  - Small control variation

$$\mathbf{x}(t) = \mathbf{x}_N(t) + \Delta\mathbf{x}(t)$$
$$\mathbf{u}(t) = \mathbf{u}_N(t) + \Delta\mathbf{u}(t)$$



Both Paths Satisfy the Dynamic Equations



$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t)], \quad x_N\left(t_0\right) \text{ given}$$
$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad x\left(t_0\right) \text{ given}$$

Alternative notation

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t)]$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t)]$$

$$\Delta\mathbf{x}(t_0) = \mathbf{x}(t_0) - \mathbf{x}_N(t_0)$$
$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_N(t)$$
$$\Delta\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_N(t)$$

$$\Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_N(t)$$

Neighboring-Optimal Trajectories

$\mathbf{x}_N$*(t) is an optimal solution to a cost function

$$\dot{\mathbf{x}}_N*(t) = \mathbf{f}[\mathbf{x}_N*(t), \mathbf{u}_N*(t)], \quad \mathbf{x}_N(t_0) \text{ given}$$

$$J_N* = \phi\left[\mathbf{x}_N*(t_f)\right] + \int_{t_0}^{t_f} L\left[\mathbf{x}_N*(t), \mathbf{u}_N*(t)\right] dt$$

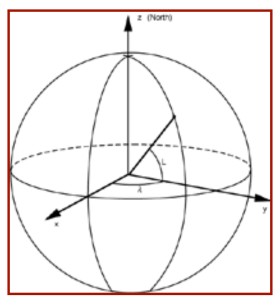
If x*(*t*) is an optimal solution to the same cost function

$$\dot{\mathbf{x}}*(t) = \mathbf{f}[\mathbf{x}*(t), \mathbf{u}*(t)], \quad \mathbf{x}(t_0) \text{ given}$$

$$J* = \phi\left[\mathbf{x}*(t_f)\right] + \int_{t_0}^{t_f} L\left[\mathbf{x}*(t), \mathbf{u}*(t)\right] dt$$

Then $\mathbf{x}_N$* and x* are neighboring-optimal trajectories

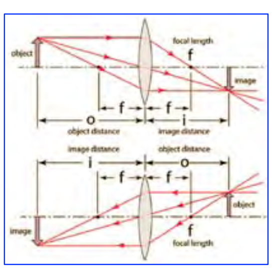
Uniqueness
Jacobi Condition



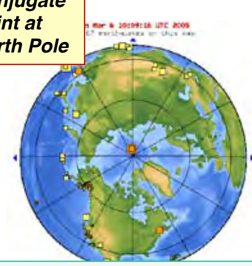
1) **Finite state perturbation implies finite control perturbation**

$$\left\{\Delta\mathbf{x}(t) < \infty\right\} \Leftrightarrow \left\{\Delta\mathbf{u}(t) < \infty\right\}$$

2) **No conjugate points (~focal points)**



Conjugate Point at North Pole



<u>Example</u>: Minimum distance from the north pole to the equator

Next Time:
Principles for Optimal Control, Part 2

Reading:
OCE: pp. 222-231

Supplemental Material

Effects of Control Weighting in Optimal Control of LTI System

$$\min_u J = \int_0^T \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + ru^2(t) \right] dt, \quad \mathbf{Q}, r > 0$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t)$$

$$\mathbf{x} = \begin{bmatrix} x_1, & \textit{displacement} \\ x_2, & \textit{rate} \end{bmatrix}$$

Example

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}, \quad a,b > 0 \;\; \text{[unstable]}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

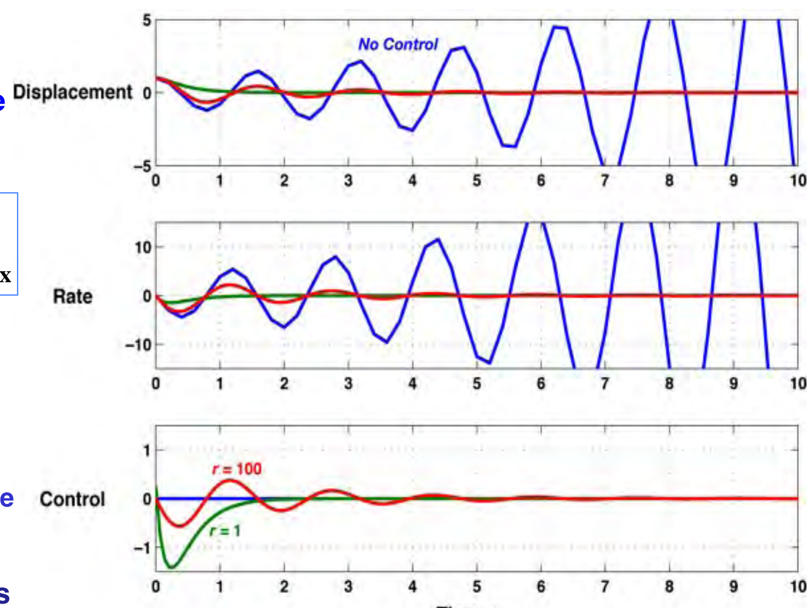
$$r = 1 \text{ or } 100$$

Effects of Control Weighting in Optimal Control of Unstable LTI System

- **Optimal feedback control *(TBD)* stabilizes unstable system response to initial condition**

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}\mathbf{x} + \mathbf{G}u_{optimal}$$
$$= \mathbf{F}\mathbf{x} - \mathbf{G}\mathbf{C}\mathbf{x} = (\mathbf{F} - \mathbf{G}\mathbf{C})\mathbf{x}$$

- **Smaller control weight**
  - **Allows larger control response**
  - **Decreases state variation**
- **Larger control weight conserves control energy**



$$\mathbf{Q} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

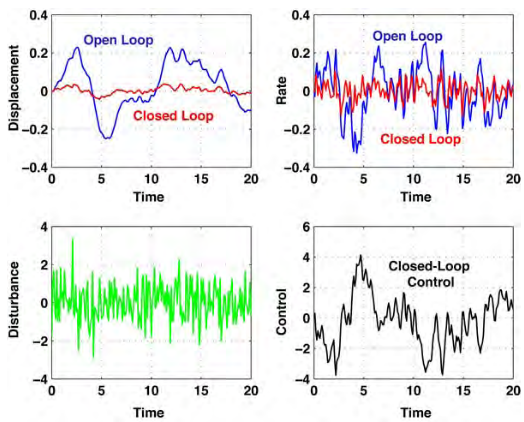
$$\mathbf{R} = 1$$

Open-Loop and Optimal Closed-Loop Response to Disturbance

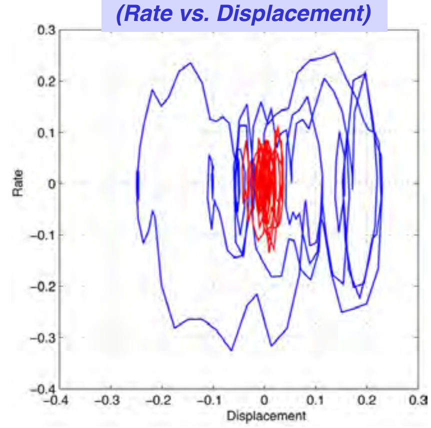
Stable 2nd-order linear dynamic system: $d\mathbf{x}(t)/dt = \mathbf{Fx}(t) + \mathbf{Gu}(t) + \mathbf{Lw}(t)$

Optimal feedback control *(TBD)* reduces response to disturbances

Time Response

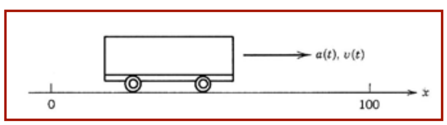


Phase-Plane Plot
(Rate vs. Displacement)



Time-Invariant Example with Scalar Control

Cart on a Track



$$H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}] = L[\mathbf{x},\mathbf{u}] + \boldsymbol{\lambda}^T \mathbf{f}[\mathbf{x},\mathbf{u}] = \text{ Constant}$$

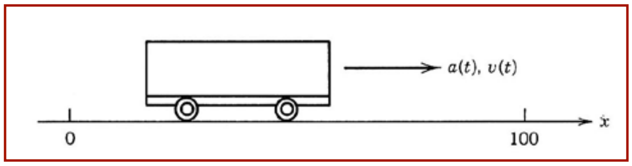
$$= ru(t)^2 + \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

$$= ru(t)^2 + \lambda_1(t)x_2(t) + \lambda_1(t)\left(t_f - t\right)u(t) = \text{ Constant}$$

$$H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}] = ru(t)^2 + \begin{bmatrix} 2q\left(x_{1_f} - 100\right) & 2q\left(x_{1_f} - 100\right)\left(t_f - t\right) \end{bmatrix} \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

$$ru(t)^2 + 2q\left(x_{1_f} - 100\right)\left(t_f - t\right)u(t) + 2q\left(x_{1_f} - 100\right)x_2(t) = \textbf{ Constant}\left(\textbf{TBD}\right)$$

Cart on a Track with Scalar Control and Open End Time



$$H^* = ru(t)^2 + \lambda_1(t)x_2(t) + \lambda_1(t)\left(t_f - t\right)u(t) = 0$$

- Fixed end-time results (t_f = 10 s)
- Open end-time would be important only if *q/r* is small

q	100	1	1
r	1	1	100
k_1	3.000	2.991	2.308
k_2	−0.300	−0.299	−0.231
x_{1_f}	99.997	99.701	76.923
x_{2_f}	15.000	14.955	11.538
$\int u^2\, dt$	29.998	29.821	17.751
J	32.794	29.923	2307.7

Examples of Open End-Time Problems

- Minimize elapsed time to achieve an objective
- Minimize fuel to go from one place to another
- Achieve final objective using a fixed amount of energy