

# Métodos de Energia

Prof. F.A.Rochinha – Julho de 2022

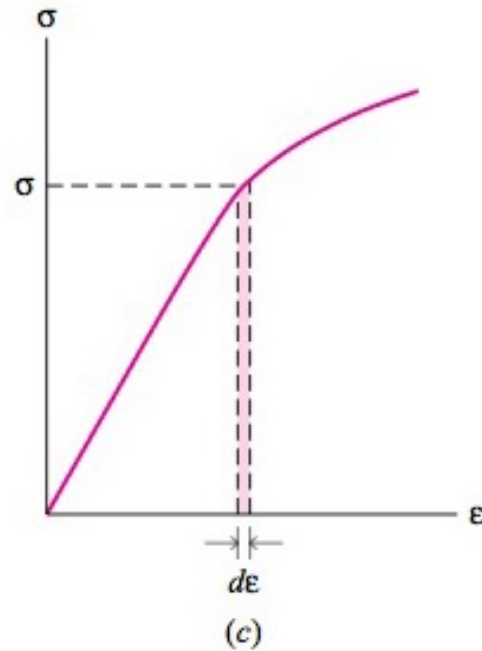
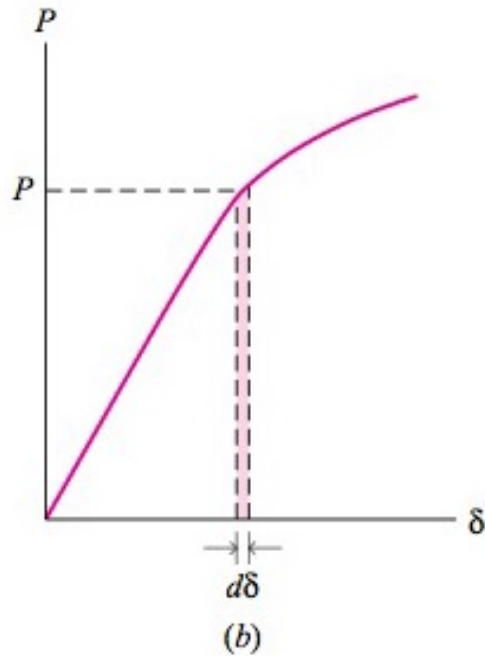
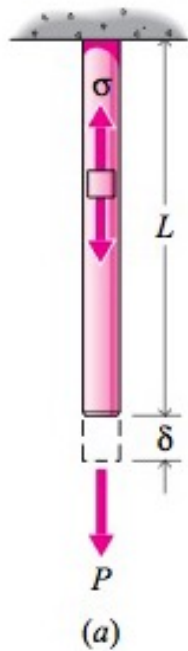
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**Ref. e ilustrações : An Introduction to the Mechanics of Solids . Crandall et. All. – 2.6, 5.8, 6.8, 7.8**



- O caso tridimensional
- Energia de deformação em torção
- Energia de deformação em flexão

# Introdução : o cenário undimensional elástico linear



$$\sigma = \frac{P}{A} ; \epsilon = \frac{\sigma}{E} ; \epsilon = \frac{\delta}{L}$$

Trabalho da força externa

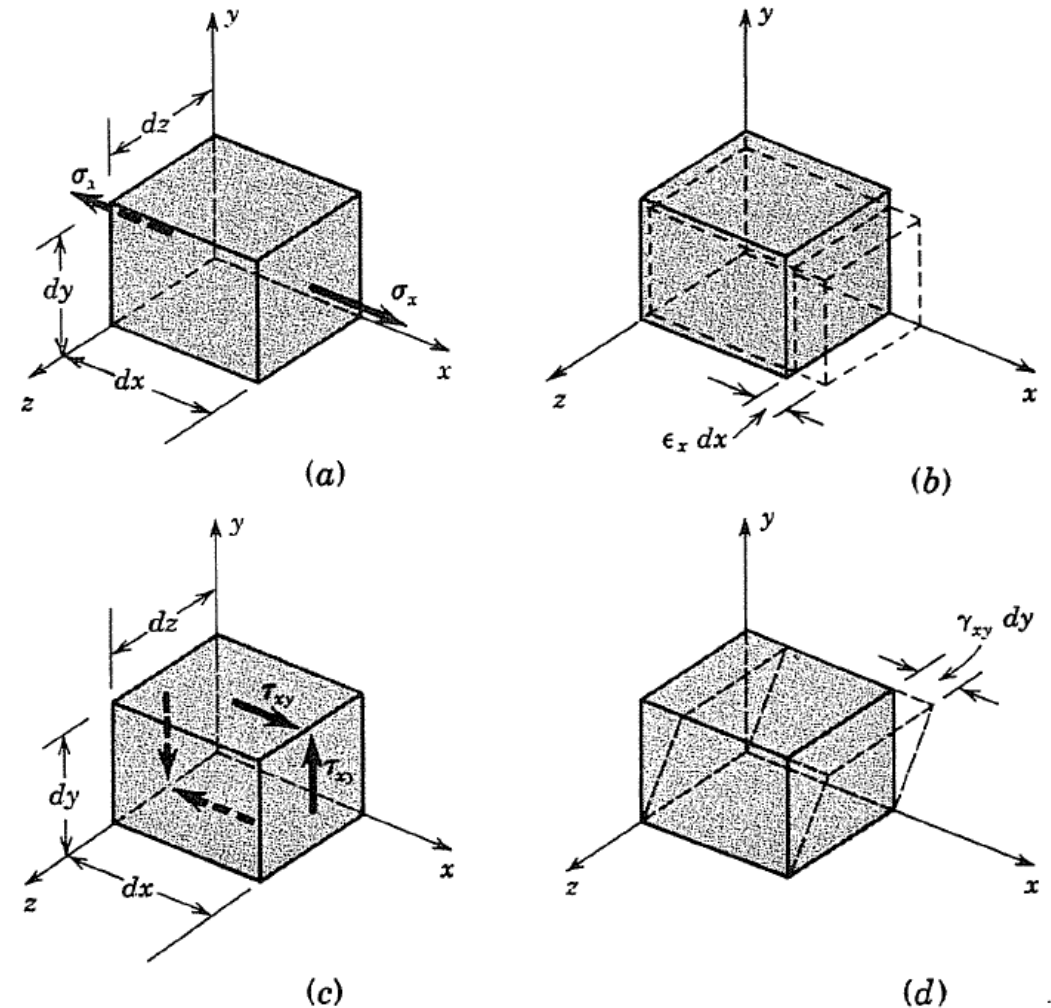
$$W = \int_0^{\delta_2} P d\delta = \int_0^{\delta_2} \sigma A d\delta = \int_0^{\epsilon_2} \sigma AL d\epsilon$$

Balanço de energia

$$U = W = \frac{\sigma^2 AL}{2E} = \frac{P^2 L}{2EA}$$

$$U^* = \frac{\epsilon^2 ALE}{2} = \frac{\delta^2 AE}{2L}$$

# Generalizando: o caso tridimensional



**Fig. 5.20** Infinitesimal element subjected to: uniaxial tension (a), with resulting deformation (b); pure shear (c), with resulting deformation (d).

Para um elemento infinitesimal e admitindo uma relação linear entre tensão e deformação ...

$$dU = \frac{1}{2}(\sigma_x \, dy \, dz) \, \epsilon_x \, dx = \frac{1}{2} \sigma_x \, \epsilon_x \, dV$$

$$dU = \frac{1}{2} \{ \sigma_x \, \epsilon_x + \sigma_y \, \epsilon_y + \sigma_z \, \epsilon_z \} dV$$

$$dU = \frac{1}{2}(\tau_{xy} \, dx \, dz) \, \gamma_{xy} \, dy = \frac{1}{2} \tau_{xy} \, \gamma_{xy} \, dV$$

$$U = \frac{1}{2} \int_V \sigma_x \, \epsilon_x + \sigma_y \, \epsilon_y + \sigma_z \, \epsilon_z + \tau_{xy} \, \gamma_{xy} + \tau_{xz} \, \gamma_{xz} + \tau_{yz} \, \gamma_{yz} \} dV$$

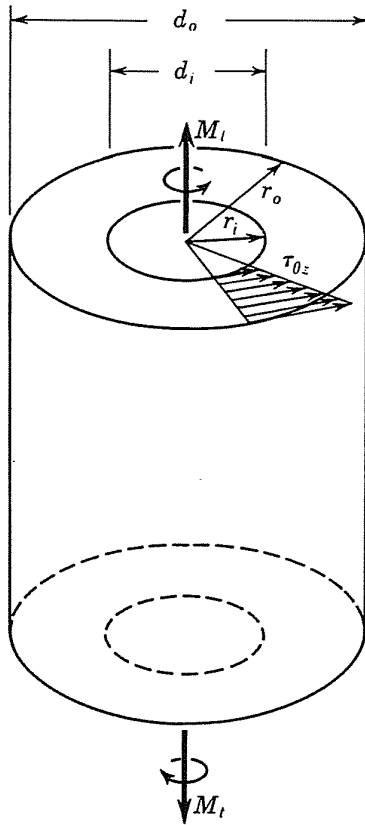
# No caso de materiais isotrópicos

$$U = \int \left[ \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E} (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + \frac{1}{2G} (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] dV$$

— termo de deformação

$$U = \int \left\{ \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)} (\epsilon_x + \epsilon_y + \epsilon_z)^2 - \frac{E}{(1+\nu)} [\epsilon_x \epsilon_y + \epsilon_y \epsilon_z + \epsilon_z \epsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)] \right\} dV$$

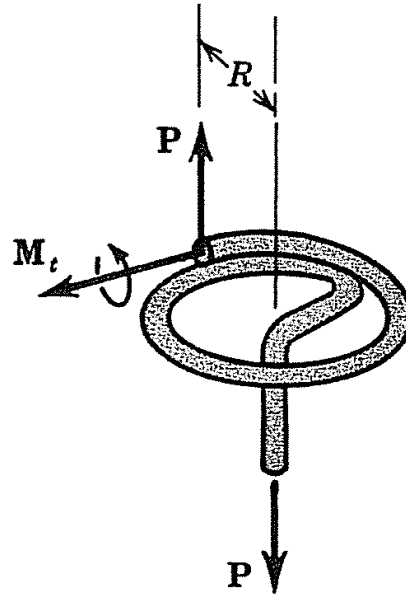
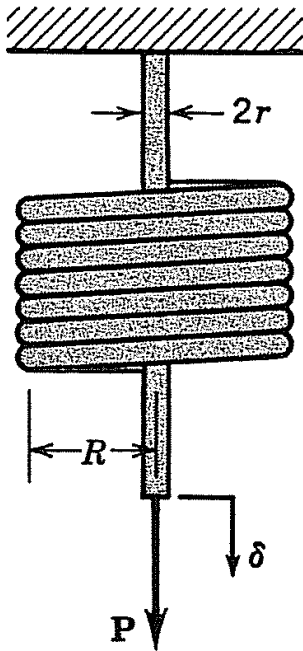
# Das expressões anteriores: Torção



$$U = \frac{1}{2} \int_V \tau_{\theta z} \gamma_{\theta z} dV$$

$$U = \frac{1}{2} \int_V \frac{1}{G} \left[ \frac{M_t r}{I_z} \right]^2 dV = \frac{1}{2} \int_L \frac{M_t^2}{GI_z^2} dz \int_A r^2 dA$$

$$U = \int_L \frac{M_t^2}{2GI_z} dz$$



$$U = \int_L \frac{P^2 R^2}{2GI_z} dz = \int_0^{2\pi n} \frac{P^2 R^2}{2GI_z} R d\theta = \frac{P^2 R^3}{2GI_z} 2\pi n$$

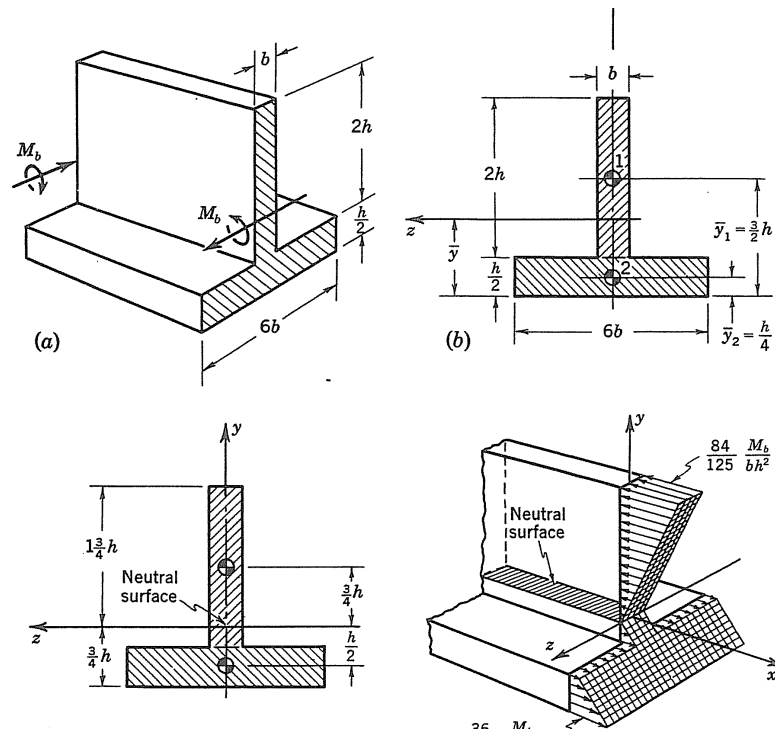
$$\delta = \frac{\partial U}{\partial P} = \frac{PR^3}{GI_z} 2\pi n$$

$$k = \frac{P}{\delta} = \frac{GI_z}{2\pi n R^3}$$

$$k = \frac{Gr^4}{4nR^3}$$



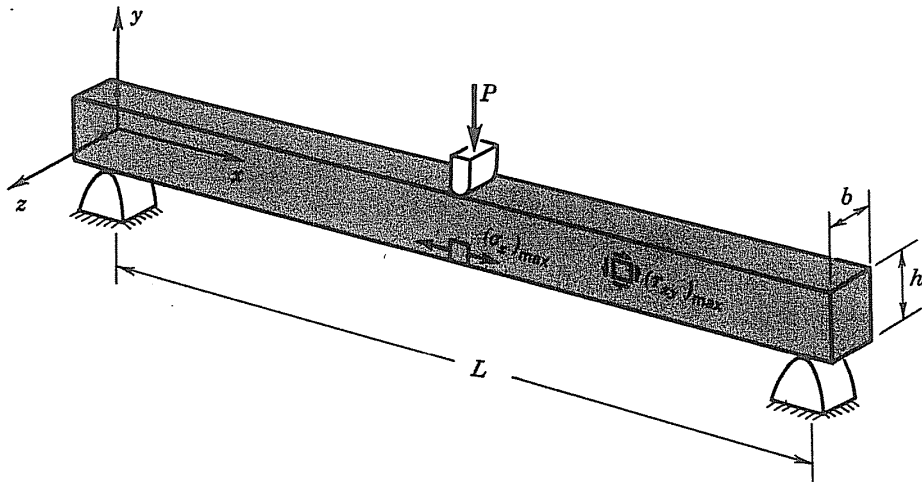
# Das expressões anteriores: Flexão



$$U = \frac{1}{2} \iiint \sigma_x \epsilon_x dx dy dz = \iiint \frac{\sigma_x^2}{2E} dx dy dz$$

$$U = \iiint \frac{1}{2E} \left( \frac{M_b y}{I_{zz}} \right)^2 dx dy dz = \int_L \frac{M_b^2}{2EI_{zz}^2} dx \int_A y^2 dy dz$$

$$U = \int_L \frac{M_b^2}{2EI_{zz}} dx$$

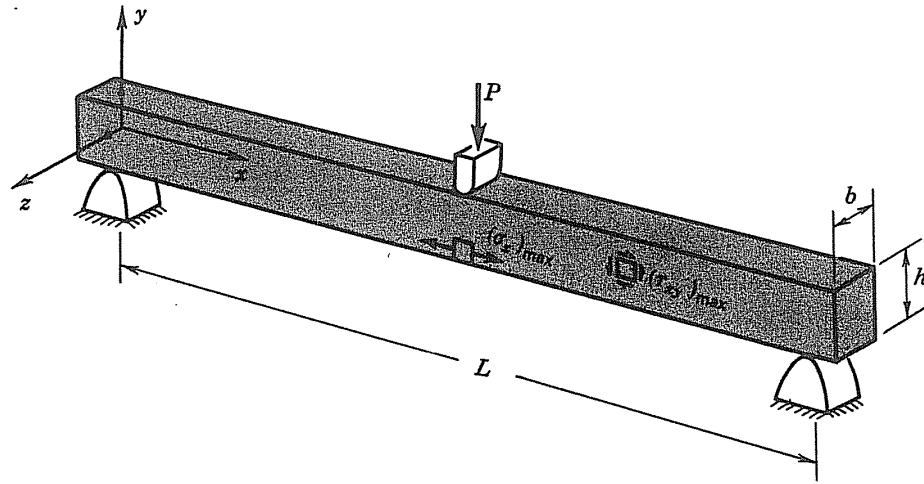


$$M_b(x) = P(x/2 - \langle x - L/2 \rangle^1)$$

$$U_b = \int_0^L \frac{M_b^2}{2EI_{zz}} dx = 2 \int_0^{L/2} \frac{(Px/2)^2}{2EI_{zz}} dx$$

Carrying out the integration yields

$$U_b = \frac{P^2 L^3}{96EI_{zz}} = \frac{P^2 L^3}{8Eb h^3}$$



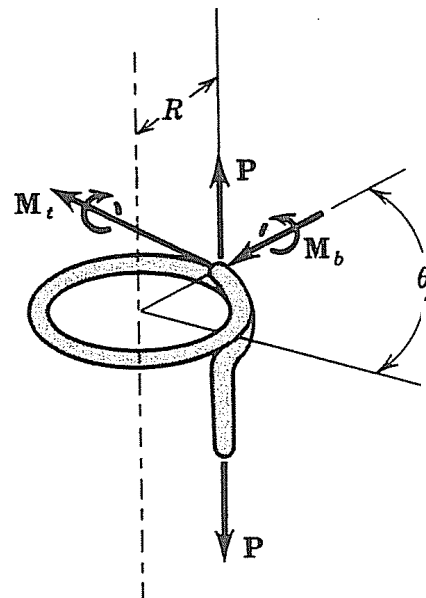
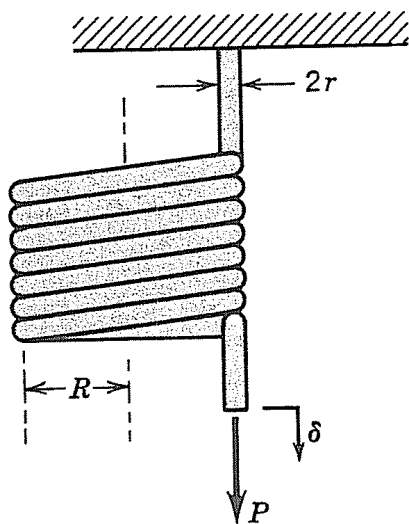
$$U = U_b + U_s = \frac{P^2 L^3}{8 E b h^3} \left[ 1 + \frac{6}{5} \frac{E}{G} \left( \frac{h}{L} \right)^2 \right]$$

$$V(x) = P(-1/2 + \langle x - L/2 \rangle^0)$$

$$\tau_{xy} = \frac{V(x)}{2 I_{zz}} \left[ \left( \frac{h}{2} \right)^2 - y^2 \right]$$

$$\frac{U_s}{U_b} = \frac{6}{5} \frac{E}{G} \left( \frac{h}{L} \right)^2 = \frac{12}{5} (1 + \nu) \left( \frac{h}{L} \right)^2$$

$$\begin{aligned} U_s &= \int_0^L \frac{V^2 dx}{8 G I_{zz}^2} \int_{-h/2}^{h/2} \left[ \left( \frac{h}{2} \right)^2 - y^2 \right]^2 dy \int_{-b/2}^{b/2} dz \\ &= \frac{P^2 L b h^5}{960 G I_{zz}^2} = \frac{3}{20} \frac{P^2 L}{G b h} \end{aligned}$$



$$U = \int_0^{2\pi n} \frac{P^2 R^2 (1 - \cos \theta)^2}{2GI_x} R d\theta + \int_0^{2\pi n} \frac{P^2 R^2 \sin^2 \theta}{2EI} R d\theta$$

$$= \frac{P^2 R^3}{2GI_x} 3\pi n + \frac{P^2 R^3}{2EI} \pi n$$

$$M_t = PR(1 - \cos \theta) \quad M_b = PR \sin \theta$$

$$\delta = \frac{\partial U}{\partial P} = PR^3 \pi n \left( \frac{3}{GI_x} + \frac{1}{EI} \right)$$

$$= \frac{4PR^3 n}{Gr^4} \left( \frac{3}{2} + \frac{G}{E} \right)$$

$$= \frac{4PR^3 n}{Gr^4} \left( \frac{4 + 3\nu}{2 + 2\nu} \right)$$