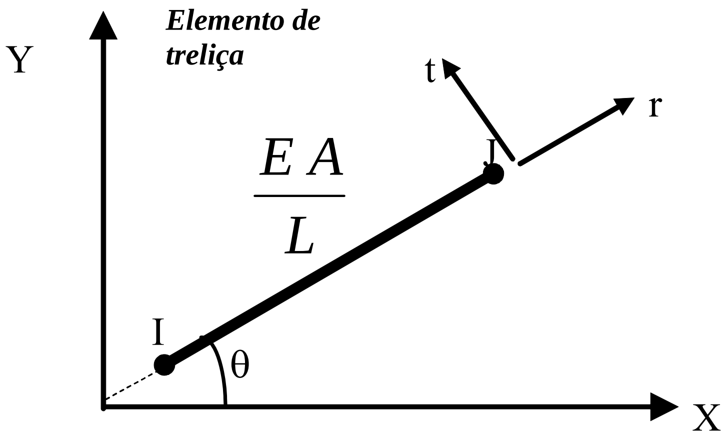
Análise Matricial de Estruturas

Análise Linear Elástica

Objetivo:
- Apresentar a estrutura matemática de um programa de elementos finitos.
- Discutir alguns aspectos gerais na programação do método dos elementos finitos



E – Módulo de Elasticidade
A – Área da Seção Transversal
L – Tamanho do Elemento

<u>Deslocamentos Nodais</u>

Sistema Local:

$$U_{eL} = \begin{pmatrix} u_r^I \\ u_t^I \\ u_r^J \\ u_t^J \end{pmatrix}$$

Sistema Global:

$$U_e = \begin{pmatrix} u_x^I \\ u_y^I \\ u_x^J \\ u_y^J \end{pmatrix}$$

$$R = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$U_{eL} = R \cdot U_e$$

$$U_e = R^T \cdot U_{eL}$$

Deformações

Alongamento das barras: $\delta_e = u_r^J - u_r^I$ ou na forma matricial

$$U_{eL} = R \cdot U_e$$

Operador de deformação: B_e

$$\delta_e = (-1 \;\; 0 \;\; 1 \;\; 0) \cdot \begin{pmatrix} u_r^I \\ u_t^I \\ u_r^J \\ u_t^J \end{pmatrix}$$

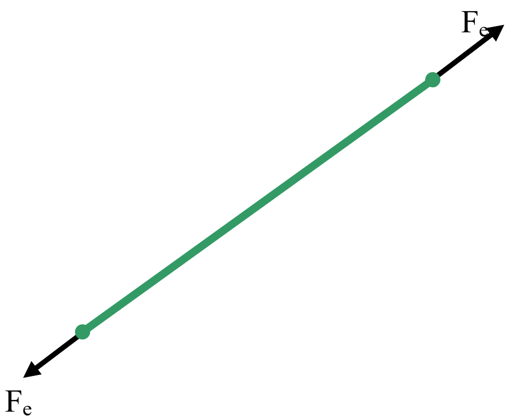
$$\delta_e = B_{eL} \cdot U_{eL} = B_e \cdot U_e \qquad B_e = B_{eL} \cdot R$$

$$B_e = (-1 \;\; 0 \;\; 1 \;\; 0) \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \rightarrow 0$$

Relação Constitutiva - Elasticidade linear :

$$F_e = \frac{E \cdot A}{L} \cdot \delta_e = \frac{E \cdot A}{L} \cdot B_e \cdot U_e$$

Esforço Interno : F_e



Equilíbrio : Princípio dos trabalhos virtuais

Treliça - forças nodais **V** - Conjunto de nós com restrições essenciais

Número de barras : *nel* F - Forças nodais
Número de nós: *nno* U - Deslocamentos Nodais

$$\sum_{n=1}^{nel} \left(F_e \cdot \delta_{ev}\right) = F \cdot U_v \quad \text{para todo} \quad U_v \quad \text{em } V \qquad \frac{E \cdot A}{L} \cdot B_e \cdot U_e$$

Potência Interna: $$\sum_{n=1}^{nel} \left(F_e \cdot \delta_{ev}\right) = \sum_{n=1}^{nel} \left[\frac{E \cdot A}{L} \cdot B_e \cdot U_e \cdot \left(B_e \cdot U_{ev}\right)\right] = \sum_{n=1}^{nel} \left[\left(B_e^T \cdot \frac{E \cdot A}{L} \cdot B_e\right) \cdot U_e \cdot U_{ev}\right]$$

$$\sum_{n=1}^{nel} \left(F_e \cdot \delta_{ev}\right) = \sum_{n=1}^{nel} \left[\left(K_e \cdot U_e\right) \cdot U_{ev}\right]$$

K_e - Matriz de rigidez elementar

$$K_e = \frac{E \cdot A}{L} \cdot (-\cos(\theta) \;\; -\sin(\theta) \;\; \cos(\theta) \;\; \sin(\theta))^T \cdot (-\cos(\theta) \;\; -\sin(\theta) \;\; \cos(\theta) \;\; \sin(\theta))$$

$$K_e = \frac{E \cdot A}{L} \cdot \begin{pmatrix} \cos(\theta)^2 & \cos(\theta)\cdot\sin(\theta) & -\cos(\theta)^2 & -\cos(\theta)\cdot\sin(\theta) \\ \cos(\theta)\cdot\sin(\theta) & \sin(\theta)^2 & -\cos(\theta)\cdot\sin(\theta) & -\sin(\theta)^2 \\ -\cos(\theta)^2 & -\cos(\theta)\cdot\sin(\theta) & \cos(\theta)^2 & \cos(\theta)\cdot\sin(\theta) \\ -\cos(\theta)\cdot\sin(\theta) & -\sin(\theta)^2 & \cos(\theta)\cdot\sin(\theta) & \sin(\theta)^2 \end{pmatrix}$$

Montagem da Matriz Global: $neq = 2 \cdot nno - nap$ nap = número de direções restritas

$$U = \begin{pmatrix} U_1 \\ U_2 \\ . \\ . \\ U_{neq} \end{pmatrix}$$

Relação entre U e U_e $U_e = L_e \cdot U$ $L_e = (4x \quad neq)$

$i := 1 .. 4$

$j = 1 .. nno$

$L_{e_{i,j}} = 1$ se G.L i do elemento é coincide com o grau de liberdade j da estrutura.

$L_{e_{i,j}} = 0$ em caso contrário

Substituindo na expressão do princípio das potências virtuais:

$$\sum_{n=1}^{nel} \left[\left(K_e \cdot U_e \right) \cdot U_{ev} \right] = P \cdot U_v$$

$$\left[\sum_{n=1}^{nel} \left(L_e{}^T \cdot K_e \cdot L_e \right) \right] \cdot U \cdot U_v = P \cdot U_v$$

$$K = \sum_{n=1}^{nel} \left(L_e{}^T \cdot K_e \cdot L_e \right)$$

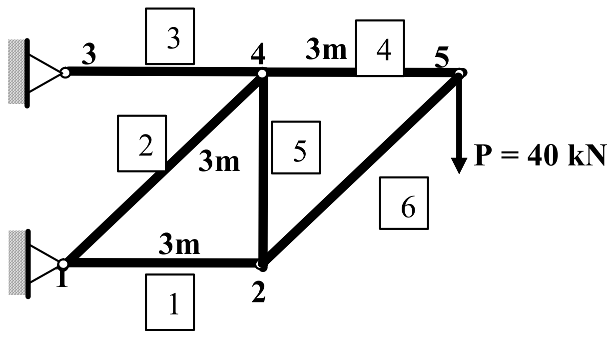
Obs : operação simbólica, não é realizada computacionalmente.

$(KU - F) \cdot U_v = 0$ para todo U_v em V

ou equivalentemente:

$$K \cdot U = F$$

Exemplo de treliça 1



Origem : nó 1

Entrada de dados:

1 - Geometria

Coordenadas : $nno := 5$ Elementos $nel := 6$

$$Coor := \begin{pmatrix} 0 & 3 & 0 & 3 & 6 \\ 0 & 0 & 3 & 3 & 3 \end{pmatrix} \qquad Inci := \begin{pmatrix} 1 & 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & 4 & 5 & 4 & 5 \end{pmatrix}$$

2 - Propriedades

$$\begin{pmatrix} E_e \\ A_e \end{pmatrix} \qquad Prop := \begin{pmatrix} 1000000 & 1000000 & 1000000 & 1000000 & 1000000 & 1000000 \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \end{pmatrix}$$

3 - Condições de contorno

$ID = (nno \;\; x \;\; 2)$

1 - preso
0 - livre

$$ID := \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} IDM \\ neq \end{pmatrix} :=$

```
neq ← 0
for i ∈ 1..nno
  for j ∈ 1..2
    aux ← ID_{i,j}
    if aux = 0
      neq ← neq + 1
      ID_{i,j} ← neq
    ID_{i,j} ← 0  otherwise
(ID, neq)
```

$$IDM = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad neq = 6$$

4 - Leitura do vetor de carga

$R := 0$

$$F_{ext} := \begin{pmatrix} R & 0 & R & 0 & 0 \\ R & 0 & R & 0 & -40 \end{pmatrix}$$

Cálculo das Matrizes Elementares:

$$K_e(e) := \left|\begin{array}{l} E \leftarrow Prop_{1,e} \\ A \leftarrow Prop_{2,e} \\ noI \leftarrow Inci_{1,e} \\ noJ \leftarrow Inci_{2,e} \\ xI \leftarrow Coor_{1,noI} \\ xJ \leftarrow Coor_{1,noJ} \\ yI \leftarrow Coor_{2,noI} \\ yJ \leftarrow Coor_{2,noJ} \\ L \leftarrow \sqrt{(xJ - xI)^2 + (yJ - yI)^2} \\ \cos \leftarrow \dfrac{xJ - xI}{L} \\ \sin \leftarrow \dfrac{yJ - yI}{L} \\ K_e \leftarrow \dfrac{E \cdot A}{L} \cdot \begin{pmatrix} \cos^2 & \cos\cdot\sin & -\cos^2 & -\cos\cdot\sin \\ \cos\cdot\sin & \sin^2 & -\cos\cdot\sin & -\sin^2 \\ -\cos^2 & -\cos\cdot\sin & \cos^2 & \cos\cdot\sin \\ -\cos\cdot\sin & -\sin^2 & \cos\cdot\sin & \sin^2 \end{pmatrix} \\ K_e \end{array}\right.$$

$$K_e(1) = \begin{pmatrix} 3.333\times10^3 & 0 & -3.333\times10^3 & 0 \\ 0 & 0 & 0 & 0 \\ -3.333\times10^3 & 0 & 3.333\times10^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_e(1) = \begin{pmatrix} 3.333\times10^3 & 0 & -3.333\times10^3 & 0 \\ 0 & 0 & 0 & 0 \\ -3.333\times10^3 & 0 & 3.333\times10^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_e(6) = \begin{pmatrix} 1.179\times10^3 & 1.179\times10^3 & -1.179\times10^3 & -1.179\times10^3 \\ 1.179\times10^3 & 1.179\times10^3 & -1.179\times10^3 & -1.179\times10^3 \\ -1.179\times10^3 & -1.179\times10^3 & 1.179\times10^3 & 1.179\times10^3 \\ -1.179\times10^3 & -1.179\times10^3 & 1.179\times10^3 & 1.179\times10^3 \end{pmatrix}$$

Matrizes Booleanas

$nnoel := 2 \qquad ngln := 2$

$$IDM = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$L_e(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Le(e) := | for j ∈ 1 .. nnoel
         |   | no ← Inci(j,e)
         |   | for i ∈ 1 .. ngln
         |   |   | eq ← IDM(no,i)
         |   |   | LM(i,j) ← eq
         | ngle ← 0
         | for j ∈ 1 .. nnoel
         |   for i ∈ 1 .. ngln
         |     | eq ← LM(i,j)
         |     | ngle ← ngle + 1
         |     | for ng ∈ 1 .. neq
         |     |   | L(ngle,ng) ← 0
         |     |   | L(ngle,eq) ← 1  if eq ≠ 0
         | L
```

$$L_e(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad L_e(3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad L_e(5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$L_e(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad L_e(4) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad L_e(6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Montagem de vetores e matrizes globais:

```
F := | for no ∈ 1 .. nno
     |   for ng ∈ 1 .. 2
     |     | eq ← IDM(no,ng)
     |     | F(eq) ← Fext(ng,no)  if eq ≠ 0
     | F
```

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -40 \end{pmatrix}$$

Montagem Simbólica - de acordo com a teoria

$$K := \left[\sum_{e=1}^{nel} \left(L_e(e)^T \cdot K_e(e) \cdot L_e(e) \right) \right]$$

$$K = \begin{pmatrix} 4.512\times10^{3} & 1.179\times10^{3} & 0 & 0 & -1.179\times10^{3} & -1.179\times10^{3} \\ 1.179\times10^{3} & 4.512\times10^{3} & 0 & -3.333\times10^{3} & -1.179\times10^{3} & -1.179\times10^{3} \\ 0 & 0 & 7.845\times10^{3} & 1.179\times10^{3} & -3.333\times10^{3} & 0 \\ 0 & -3.333\times10^{3} & 1.179\times10^{3} & 4.512\times10^{3} & 0 & 0 \\ -1.179\times10^{3} & -1.179\times10^{3} & -3.333\times10^{3} & 0 & 4.512\times10^{3} & 1.179\times10^{3} \\ -1.179\times10^{3} & -1.179\times10^{3} & 0 & 0 & 1.179\times10^{3} & 1.179\times10^{3} \end{pmatrix}$$

Montagem computacionalmente eficiente

```
K := | for i ∈ 1 .. neq
     |   for j ∈ 1 .. neq
     |     K[i,j] ← 0
     | for e ∈ 1 .. nel
     |   | for j ∈ 1 .. nnoel
     |   |   | no ← Inci[j,e]
     |   |   | for i ∈ 1 .. ngln
     |   |   |   | eq ← IDM[no,i]
     |   |   |   | LM[i,j] ← eq
     |   | for j ∈ 1 .. nnoel
     |   |   | ii ← 2 · (j − 1)
     |   |   | for i ∈ 1 .. ngln
     |   |   |   | l ← LM[i,j]
     |   |   |   | ll ← ii + i
     |   |   |   | for n ∈ 1 .. nnoel                          if l ≠ 0
     |   |   |   |   | jj ← 2 · (n − 1)
     |   |   |   |   | for k ∈ 1 .. ngln
     |   |   |   |   |   | m ← LM[k,n]
     |   |   |   |   |   | ml ← jj + k
     |   |   |   |   |   | K[l,m] ← K[l,m] + Ke(e)[ll,ml]   if m ≠ 0
     | K
```

$$K = \begin{pmatrix} 4.512\times10^{3} & 1.179\times10^{3} & 0 & 0 & -1.179\times10^{3} & -1.179\times10^{3} \\ 1.179\times10^{3} & 4.512\times10^{3} & 0 & -3.333\times10^{3} & -1.179\times10^{3} & -1.179\times10^{3} \\ 0 & 0 & 7.845\times10^{3} & 1.179\times10^{3} & -3.333\times10^{3} & 0 \\ 0 & -3.333\times10^{3} & 1.179\times10^{3} & 4.512\times10^{3} & 0 & 0 \\ -1.179\times10^{3} & -1.179\times10^{3} & -3.333\times10^{3} & 0 & 4.512\times10^{3} & 1.179\times10^{3} \\ -1.179\times10^{3} & -1.179\times10^{3} & 0 & 0 & 1.179\times10^{3} & 1.179\times10^{3} \end{pmatrix}$$

Solução do Sistema:

$U := K^{-1} F$

$$U = \begin{pmatrix} -0.012 \\ -0.07 \\ 0.024 \\ -0.058 \\ 0.036 \\ -0.152 \end{pmatrix}$$

Pós-Processamento:

```
δU := | for no ∈ 1 .. nno
      |   for ng ∈ 1 .. 2
      |     | V[no,ng] ← 0
      |     | eq ← IDM[no,ng]
      |     | V[no,ng] ← U[eq]  if eq ≠ 0
      | V
```

$$\delta U = \begin{pmatrix} 0 & 0 \\ -0.012 & -0.07 \\ 0 & 0 \\ 0.024 & -0.058 \\ 0.036 & -0.152 \end{pmatrix}$$

$i := 1 .. nno$

$x_i := Coor_{1,i}$ $\quad y_i := Coor_{2,i}$

$$x = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 6 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} \quad Inci = \begin{pmatrix} 1 & 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & 4 & 5 & 4 & 5 \end{pmatrix}$$

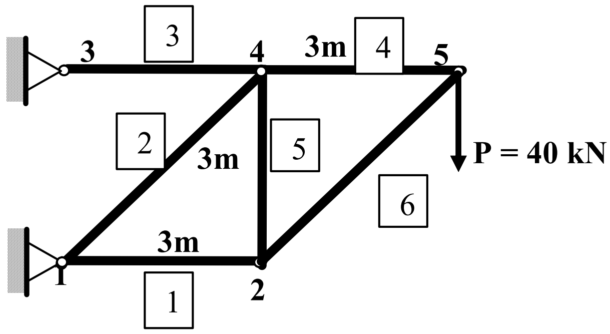
$xn_i := x_i + 3 \cdot \delta U_{i,1}$ $\quad yn_i := y_i + 3 \cdot \delta U_{i,2}$

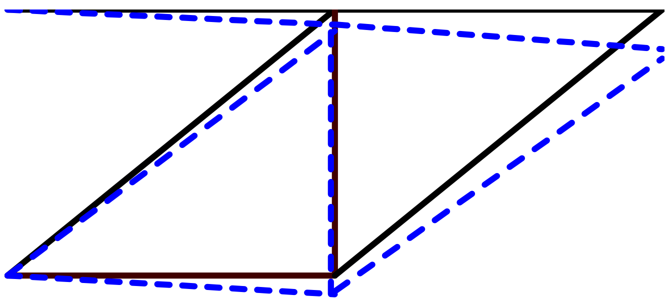
$$xn = \begin{pmatrix} 0 \\ 2.964 \\ 0 \\ 3.072 \\ 6.108 \end{pmatrix} \quad yn = \begin{pmatrix} 0 \\ -0.21 \\ 3 \\ 2.826 \\ 2.544 \end{pmatrix}$$

Deformada

$s1 := 0, 3 .. 3$

$s2 := 3, 6 .. 6$





$$
\text{Internos}(e) :=
\begin{array}{|l}
E \leftarrow Prop_{1,e} \\
A \leftarrow Prop_{2,e} \\
noI \leftarrow Inci_{1,e} \\
noJ \leftarrow Inci_{2,e} \\
xI \leftarrow Coor_{1,noI} \\
xJ \leftarrow Coor_{1,noJ} \\
yI \leftarrow Coor_{2,noI} \\
yJ \leftarrow Coor_{2,noJ} \\
L \leftarrow \sqrt{(xJ - xI)^2 + (yJ - yI)^2} \\
\cos \leftarrow \dfrac{xJ - xI}{L} \\
\sin \leftarrow \dfrac{yJ - yI}{L} \\
\delta_e \leftarrow (-\cos \;\; -\sin \;\; \cos \;\; \sin) \cdot \begin{pmatrix} \delta U_{noI,1} \\ \delta U_{noI,2} \\ \delta U_{noJ,1} \\ \delta U_{noJ,2} \end{pmatrix} \\
F_e \leftarrow \dfrac{E \cdot A}{L} \cdot \delta_e \\
\left(\delta_e \;\; F_e\right)
\end{array}
$$

$\delta_e(e) := \text{Internos}(e)_{1,1}$

$F_e(e) := \text{Internos}(e)_{1,2}$

$\delta_e(1) = -0.012$ $\delta_e(4) = 0.012$

$\delta_e(2) = -0.024$ $\delta_e(5) = 0.012$

$\delta_e(6) = -0.024$

$\delta_e(3) = 0.024$

$e := 1 .. nel$